INVERSE SPIN-S PORTRAIT METHOD: REPRESENTATION OF QUDIT STATES BY PROBABILITY VECTORS AND SPIN TOMOGRAPHY

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A spin-j state (qudit state) is usually described by a density operator $\hat{\rho}$. However, many alternative ways were proposed; for example, different discrete Wigner functions and a fair probability-distribution function $w^{(j)}(m,u) = \langle jm|\hat{u}^{\dagger}\hat{\rho}\hat{u}|jm\rangle$ also known as a unitary spin tomogram. Here, the spin projection m takes the values $-j,-j+1,\dots,j$, and $\hat{u}$ is, in general, a unitary transform of the group $SU(N)$. In the particular case $u\in SU(2)$ we obtain a so-called spin tomogram $w^{(j)}(m,\mathbf{n})$, where the direction $\mathbf{n}\equiv\mathbf{n}(\theta,\phi) = (\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta)$ determines a unitary transform $\hat{u}(\mathbf{n}) = \mathrm{e}^{-\mathrm{i}(\mathbf{n}_{\perp}\cdot\hat{\mathbf{J}})\theta}$ with $\mathbf{n}_{\perp} = (-\sin\phi,\cos\phi,0)$. The inverse mapping of spin tomogram onto the density operator $\hat{\rho}$ is relatively easily expressed through integration over sphere $\mathbf{n}\in S^2$. We will refer to a $(2s+1)$-dimensional probability vector $\mathbf{w}_s^{(j)}(u)$ with components $\sum_{m\in\mathcal{A}_i} w^{(j)}(m,u)$, as the spin-$s$ portrait of a qudit-j state ($\mathcal{A}_i\subset\{m\}_{-j}^{j}$ for all $i=1,\dots,2s+1$, $\cup_{i=1}^{2s+1}\mathcal{A}_i = \{m\}_{-j}^{j}$, and $\mathcal{A}_i\cap\mathcal{A}_l=\emptyset$ for all $i\neq l$). The aim of this report is to present a construction of the invertible map which provides the possibility to identify any spin state with the probability vector $\boldsymbol{\mathcal{P}}$ with components $\mathcal{P}(m,u_k)$, where u_k is a unitary matrix from a finite set of unitary matrices $\{u_k\}_{k=1}^{N_s}$. Thus, the function $\mathcal{P}(m,u_k)$ is a joint probability distribution of two random variables m and u_k. The vector $\boldsymbol{\mathcal{P}}$ is constructed by stacking a proper number N_s of spin-s portraits $\mathbf{w}_s^{(j)}(u_k)$ (inverse spin-s portrait method). Invertibility of the map $\hat{\rho}\to\boldsymbol{\mathcal{P}}$ restricts the minimal number of portraits N_s. In case $s=s_{\min}=1/2$ only one spin projection, e.g. $m=j$, is relevant and $N_s=(2j+1)^2$. This case corresponds to reconstruction procedure [1]. If $s=s_{\max}=j$, then $N_s=2j+2$ if all unitary matrices u_k are elements of the group $SU(N)$ with $N=2j+1$, and $N_s=4j+1$ if $u_k\in SU(2)$ for all k. The latter case is considered in detail in [2]. The direct map $\hat{\rho}\to\boldsymbol{\mathcal{P}}$ and the inverse map $\boldsymbol{\mathcal{P}}\to\hat{\rho}$ are presented in the explicit form for an *arbitrary* choice of directions $\{\mathbf{n}_k\}_{k=1}^{4j+1}$. When directions $\{\mathbf{n}_k\}_{k=1}^{4j+1}$ are specifically chosen to form a cone, the reconstruction procedure boils down to that proposed in [3]. The kernel of the corresponding star-product quantization scheme as well as intertwining kernels relating $\mathcal{P}$-representation and w-tomographic representation are found.

Quantum states form a convex subset on the $2j(4j+3)$-simplex of possible probability vectors $\boldsymbol{\mathcal{P}}$, with the boundary of quantum states being the $(2j+1)$-degree body of vector elements $\mathcal{P}(m,u)$. Examples of quantum subsets for qubits ($j=1/2$) and qutrits ($j=1$) are presented in [2,4,5]. Components $\mathcal{P}(m,u)$ are fair probabilities, have a clear physical meaning, and can be relatively easily measured experimentally. A subsidiary problem of the *optimum* choice of directions $\{\mathbf{n}_k\}_{k=1}^{4j+1}$ is discussed and partially solved for the low spin states, with the optimality implying the minimum relative error $\|\delta\hat{\rho}\|_2$ if errors $\delta\boldsymbol{\mathcal{P}}$ in the measured probability vector $\boldsymbol{\mathcal{P}}$ are presented.

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