# Optical homodyne tomography: operational use of data and evaluation of errors

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[M. Bellini, A.S. Coelho, S.N. Filippov, V.I. Man'ko, A. Zavatta. Phys. Rev. A **85**, 052129 (2012)]

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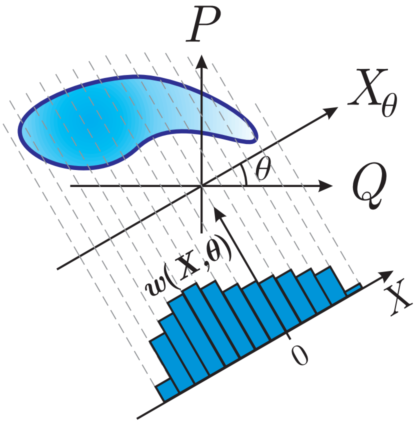
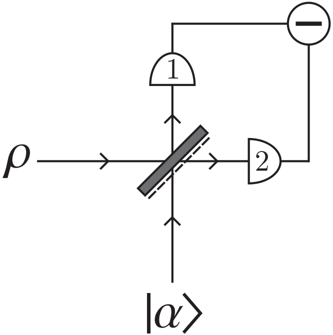
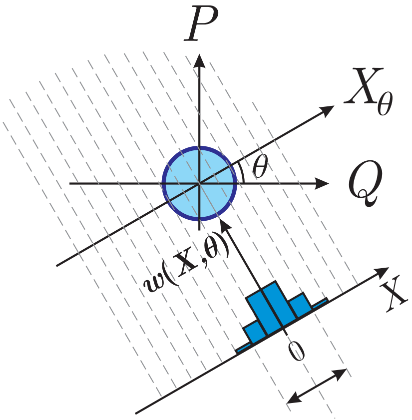
# Homodyne detection

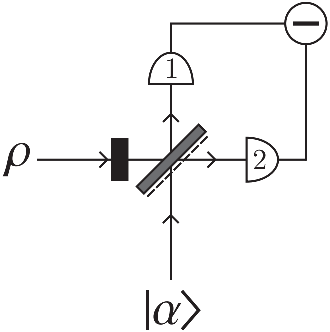



- Quadratures $\hat{X}_\theta = \hat{Q}\cos\theta + \hat{P}\sin\theta$, where $\hat{Q} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right)$ and $\hat{P} = \frac{1}{\sqrt{2}i}\left(\hat{a} - \hat{a}^\dagger\right)$
- Optical tomogram $w(X,\theta) = \langle X_\theta|\hat{\rho}|X_\theta\rangle$ contains the complete information about a quantum state

# Homodyne detection: calibration




- $\langle X \rangle = 0$
- $\sigma_{XX} = \langle (X - \langle X \rangle)^2 \rangle = \hbar/2$
- $[\hat{Q}, \hat{P}] = i\hbar$ (for example, $\hbar = \frac{1}{2}$)

# Usual approach to homodyne detection

Optical tomogram $w(X,\theta)$ for points $X \in (-\infty,+\infty)$ and $\theta \in [0,\pi)$

$$\Downarrow$$

State reconstruction
(inverse Radon transformation, pattern-functioning,
maximal likelihood approach, etc.)

$$\Downarrow$$

State characteristics

# Challenges of homodyne detection

Optical tomogram $w(X, \theta)$ for points $X \in (-\infty, +\infty)$ and $\theta \in [0, \pi)$

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State characteristics and their errors

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State characteristics and their errors (statistical and systematic)
Operational use of data

# Coherent and SPAC states

- Coherent state: $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$,

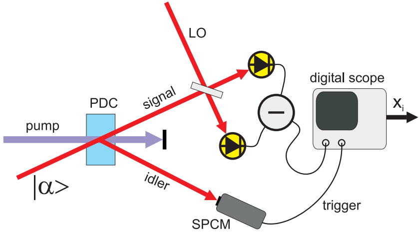
$$w_{|\alpha\rangle}(X,\theta) = \frac{1}{\sqrt{\pi\hbar}} \exp\left\{-\left[\frac{X}{\sqrt{\hbar}} - \sqrt{2}(\mathrm{Re}\alpha\cos\theta + \mathrm{Im}\alpha\sin\theta)\right]^2\right\}.$$

- Single photon added coherent state (SPACS): $\dfrac{\hat{a}^\dagger|\alpha\rangle}{\sqrt{1+|\alpha|^2}}$,

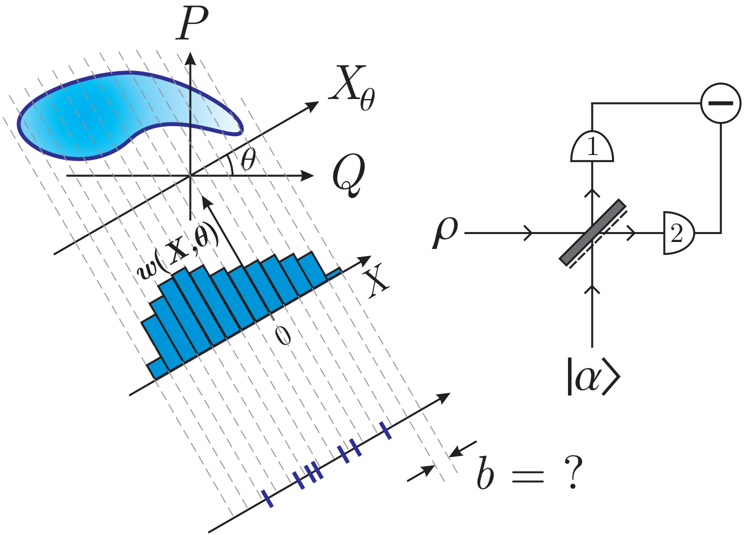
$$w_{\hat{a}^\dagger|\alpha\rangle}(X,\theta) = \left[\sqrt{\pi\hbar}(1+|\alpha|^2)\right]^{-1}$$

$$\times\left\{2\left[\frac{X}{\sqrt{\hbar}} - \frac{1}{\sqrt{2}}(\mathrm{Re}\alpha\cos\theta + \mathrm{Im}\alpha\sin\theta)\right]^2 + (\mathrm{Re}\alpha\sin\theta - \mathrm{Im}\alpha\cos\theta)^2\right\}$$

$$\times\exp\left\{-\left[\frac{X}{\sqrt{\hbar}} - \sqrt{2}(\mathrm{Re}\alpha\cos\theta + \mathrm{Im}\alpha\sin\theta)\right]^2\right\}.$$

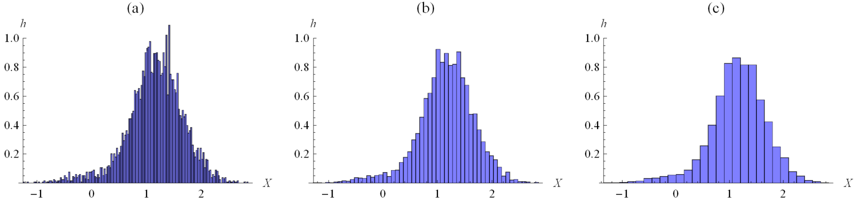


# Tomogram estimation




- Bin width = ?

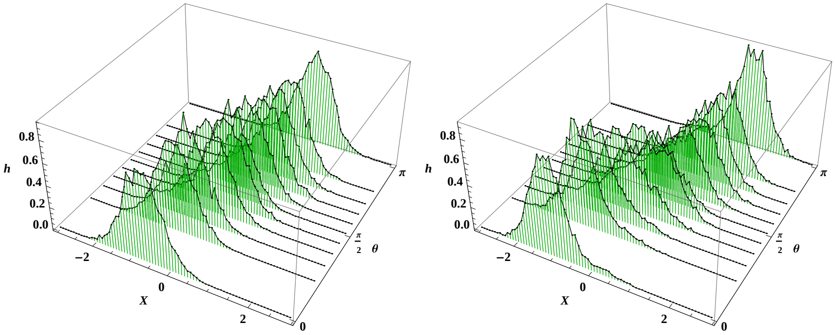
# Tomogram estimation



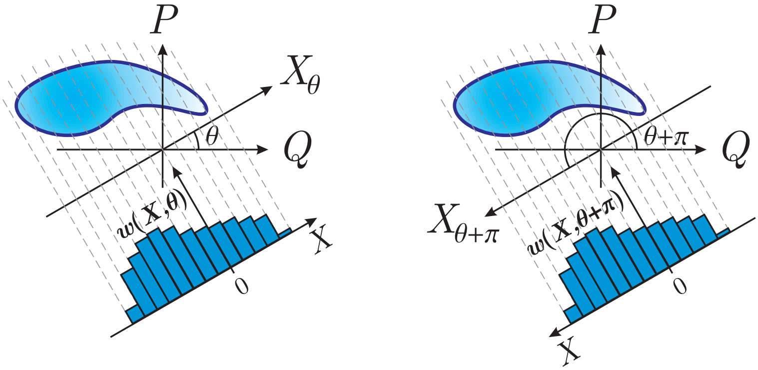

Optimal choice $b_{\text{opt}} = \left[ \frac{\pi}{4\sqrt{2}\ h(X_i, \theta_j)\ N\ d} \right]^{1/3}$

- ensures statistical confidence, error $\delta h_{\text{stat}} = \sqrt{h(X_i, \theta_j)/Nb}$
- prevents form undersampling, error $\delta h_{\text{und}} = h(X_i, \theta_j) b \sqrt{2d}/\pi$

# Coherent and SPAC states: experimental tomograms

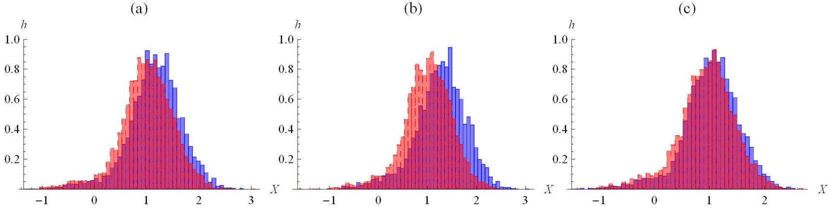


# Accuracy of optical homodyne tomograms



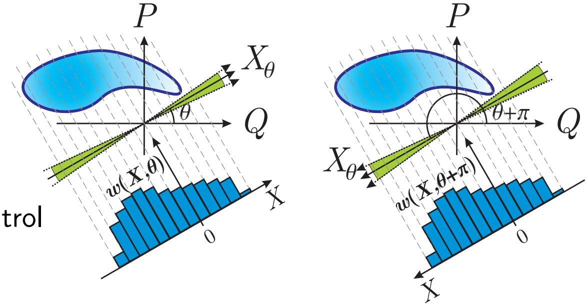

Symmetry relation $\boxed{w(X,\theta) = w(-X,\theta+\pi)}$

# Symmetry "breaking" in the experiment

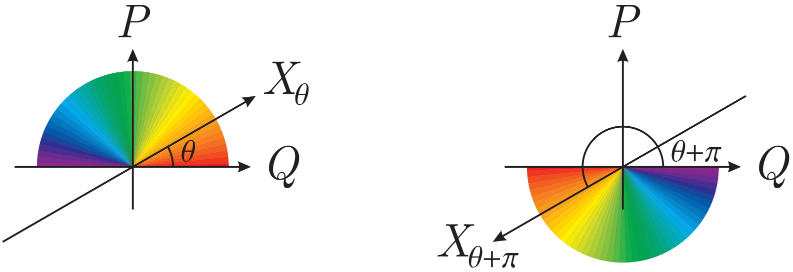



Possible reasons:

- imbalance
- inaccuracy in LO phase control
- extra noise



# Accuracy of homodyne measurement



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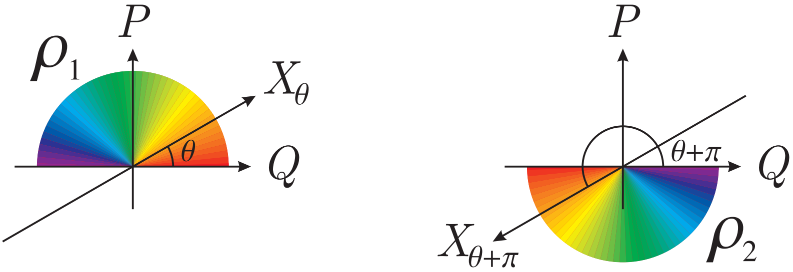


$$F(\rho_1,\rho_2)=\mathrm{Tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\leq\min_{\theta\in[0,\pi]}\int\sqrt{w(X,\theta)w(-X,\theta+\pi)}dX$$

[S.N. Filippov, V.I. Man'ko. Phys. Scr. T140, 014043 (2010)]

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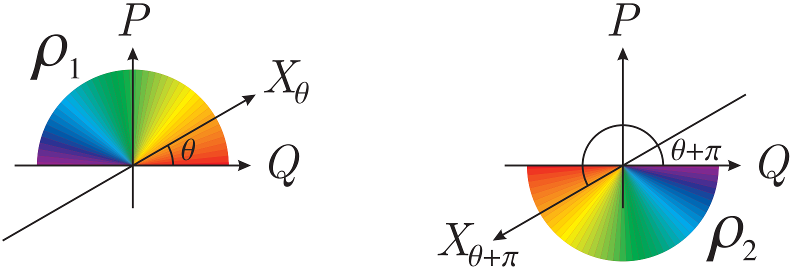


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98.32%, 95.59%, 99.26%

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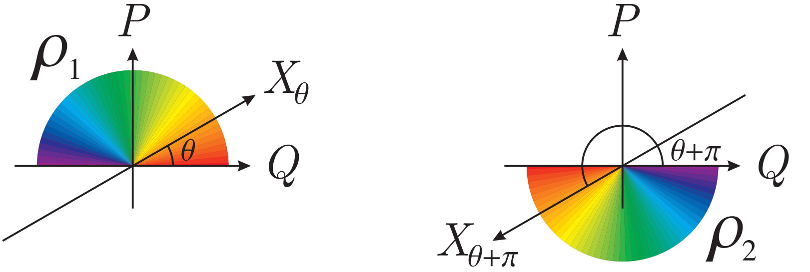


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Error of calculation: $-\frac{1}{12}b^3\frac{d^2}{dX^2}\sqrt{w(X,\theta)w(-X,\theta+\pi)} < 0.004\%$

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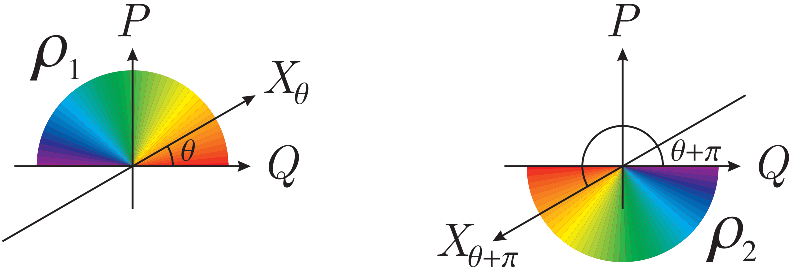


Moment $\langle X_\theta^n \rangle = \int X^n w(X,\theta) dX$ is determined with the experimental error $\Delta(X_\theta^n) = \int X^n \left| w(X,\theta) - w(-X,\theta+\pi) \right| dX$

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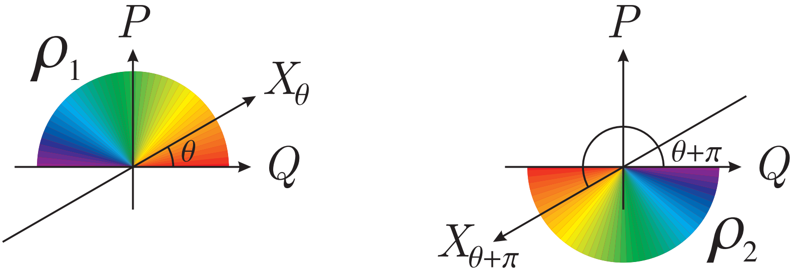


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Variances $\sigma_{X_\theta X_\theta} = \langle X_\theta^2 \rangle - \langle X_\theta \rangle^2$

0.004 (coherent) and 0.013 (SPACS)

## Purity $\mu = \mathrm{Tr}\rho^2$

$$\mu = \frac{1}{2\pi}\int_0^{+\infty} dr\, r \iint_{-\infty}^{+\infty} dX\, dY\, e^{-i(X+Y)r} \int_0^{2\pi} d\theta\, w(X,\theta) w(-Y,\theta)$$

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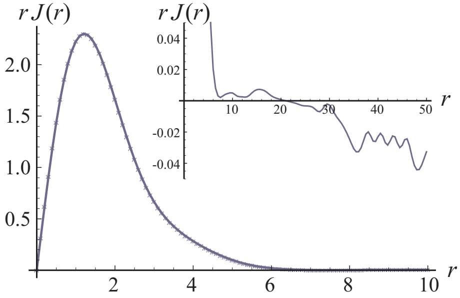
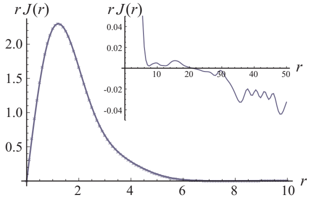


# Purity $\mu = \mathrm{Tr}\rho^2$

$$\Delta\mu = \frac{1}{2\pi}\int_0^{+\infty} r dr \iint_{-\infty}^{+\infty} dX dY \cos[(X+Y)r]\times$$
$$\times \int_0^{\pi} d\theta[w(X,\theta)w(-Y,\theta) - w(X,\theta+\pi)w(-Y,\theta+\pi)]$$

$$\Delta_{\mathrm{calc}}\mu(R) \lesssim 0.01\int_0^{+\infty} re^{-r^2/2}dr + 10^{-5}\int_0^R r(2+r^2)dr$$
$$\approx 0.01 + 3\cdot 10^{-6}R^4$$




$\mu_{\mathrm{coherent}} = 1.00 \pm 0.04$ and $\mu_{\mathrm{SPACS}} = 0.83 \pm 0.04$

# Detection imperfection

$$W^{\rm det}(q,p) = \frac{1}{\pi(1-\eta)} \iint dq'\ dp'\ W(q',p') \exp\left[-\frac{(q-\sqrt{\eta}q')^2+(p-\sqrt{\eta}p')^2}{1-\eta}\right]$$

Coherent state: $|\alpha\rangle \longrightarrow |\sqrt{\eta}\alpha\rangle$

SPACS:

$$W^{\rm det}_{\hat{a}^\dagger|\alpha\rangle}(q,p) = \left[\pi(1+|\alpha|^2)\right]^{-1}$$
$$\times\left\{1+2\eta\left[\left(q-\frac{2\eta-1}{\sqrt{2\eta}}{\rm Re}\alpha\right)^2+\left(p-\frac{2\eta-1}{\sqrt{2\eta}}{\rm Im}\alpha\right)^2-1\right]\right\}$$
$$\times\exp\left[-\left(q-\sqrt{2\eta}{\rm Re}\alpha\right)^2-\left(p-\sqrt{2\eta}{\rm Im}\alpha\right)^2\right]$$

$$\mu^{\rm det}_{\hat{a}^\dagger|\alpha\rangle} = 2\pi\iint dq\,dp\left[W^{\rm det}_{\hat{a}^\dagger|\alpha\rangle}(q,p)\right]^2 = 1-\frac{2\eta(1-\eta)}{(1+|\alpha|^2)^2}$$

For nominal values $\eta = 0.6$ and $\alpha = 0.83$ we obtain $\mu = 0.83$

# Experimental check of uncertainty relations

$$\sigma_{qq}\sigma_{pp} \geq \hbar^2\Phi^2(\mu)/4$$

$\Phi(\mu) = 2 - \sqrt{2\mu - 1}$ if $\frac{5}{9} \leq \mu \leq 1$

SPACS: $0.101 \pm 0.006 \geq 0.085 \pm 0.006$

State-extended uncertainty relation:

$$\frac{1}{2}\left(\sigma_{qq}^{(1)}\sigma_{pp}^{(2)} + \sigma_{qq}^{(2)}\sigma_{pp}^{(1)}\right) \geq \frac{\hbar^2}{4}$$

Coherent + SPACS: $0.160 \pm 0.006 > 0.0625$

# Experimental check of entropic relations

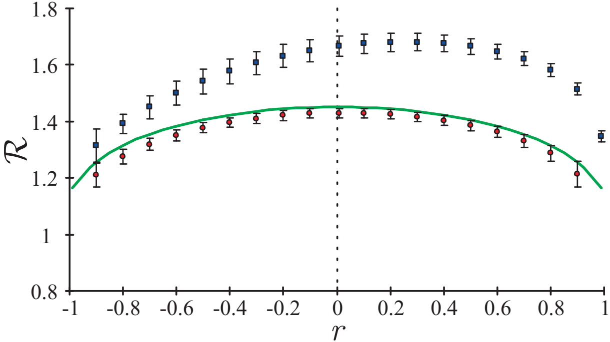
$$S(\theta) + S(\theta + \pi/2) \geq \ln(\pi\hbar) + 1$$

$$2\int_0^\pi S(\theta)\frac{d\theta}{\pi} \geq \ln(\pi\hbar) + 1$$

$$H_{X,\theta} = -\int_{-\infty}^{+\infty} dX \int_0^\pi \frac{d\theta}{\pi}\, w(X,\theta)\ln w(X,\theta) \geq \frac{1}{2}\left[\ln\pi + 1\right]$$

Coherent state: $1.42 \pm 0.01$
SPACS: $1.70 \pm 0.03$

# Rényi entropy




$$\frac{1+r}{r}\ln\left\{\int[w(X,0)]^{(1+r)^{-1}}dX\right\}-\frac{1-r}{r}\ln\left\{\int[w(X,\tfrac{\pi}{2})]^{(1-r)^{-1}}dX\right\}\geqslant$$

$$\geqslant\quad \ln(\pi\hbar)+\tfrac{1}{2r}[(1+r)\ln(1+r)-(1-r)\ln(1-r)]$$

# Summary

- Measurable quantities
- Statistical confidence and no undersampling
- Symmetry relation to evaluate accuracy
- Direct calculation of purity
- Uncertainty relations with error bars