Quantum-informational aspects of *entanglement-annihilating* processes

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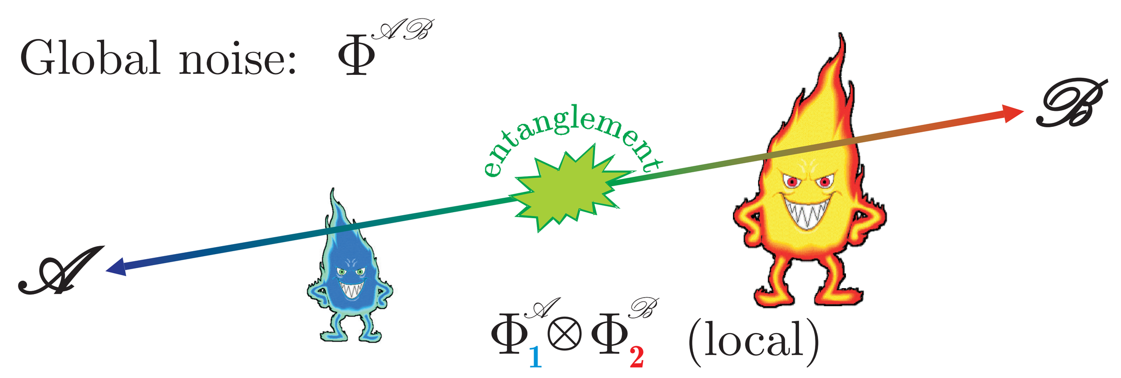
Motivation

Entanglement ⇒ numerous possible applications. The practical realization of those applications usually faces a problem of unavoidable noises and their effect on the ability to perform a desired entanglement-assisted task.

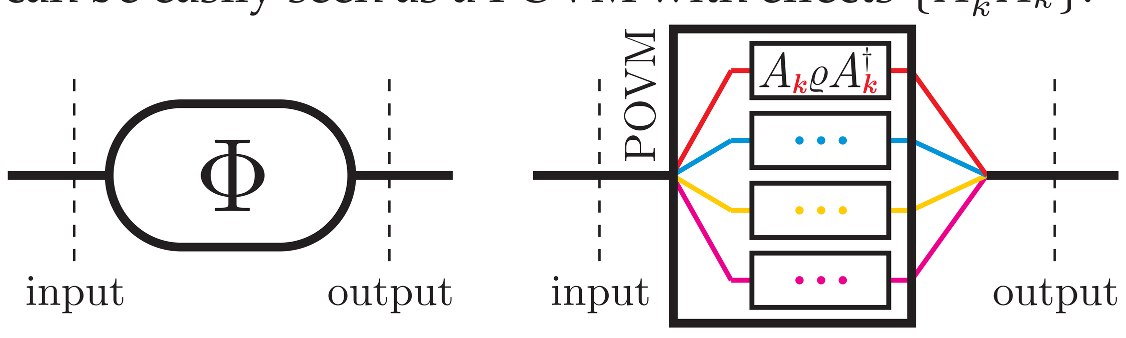


Channels that are capable to reduce the entanglement were studied for one-sided noises of the form $\Phi \otimes \mathrm{Id}$ [1]. When it happens that the output state $(\Phi^{\mathscr{A}} \otimes \mathrm{Id}^{\mathscr{R}})[\varrho_{\rm in}^{\mathscr{AR}}]$ is separable w.r.t. partition $\mathscr{A}|\mathscr{R}$ for *all* possible input states and $d^{\mathscr{A}} \le d^{\mathscr{R}}$, Φ is called *entanglement-breaking* (EB). The set of EB channels is convex and quite well studied [2].

If the state $\Phi^{\mathscr{ABC}\cdots}[\varrho_{\rm in}^{\mathscr{ABC}\cdots}]$ is separable w.r.t. partition $\mathscr{A}|\mathscr{B}|\mathscr{C}|\ldots$ for *all* input states, then the map Φ is called *entanglement-annihilating* (EA) [3].

Channel description

Sum diagonal representation $\Phi[\varrho] = \sum_k A_k \varrho A_k^\dagger$. Kraus operators $A_k : \mathcal{H}_{\rm in} \to \mathcal{H}_{\rm out}$ satisfy the trace-preserving relation $\sum_k A_k^\dagger A_k = I_{\rm in}$. The channel Φ can be easily seen as a POVM with effects $\{A_k^\dagger A_k\}$:



Choi–Jamiołkowski operator

$$\Omega_\Phi^{\mathscr{A}\mathscr{A}'} := (\Phi^{\mathscr{A}} \otimes \mathrm{Id}^{\mathscr{A}'})[|\Psi_+^{\mathscr{A}\mathscr{A}'}\rangle\langle\Psi_+^{\mathscr{A}\mathscr{A}'}|], \qquad (1)$$

where $|\Psi_+^{\mathscr{A}\mathscr{A}'}\rangle = (d^{\mathscr{A}})^{-1} \sum_{i=1}^{d^{\mathscr{A}}} |i \otimes i'\rangle$ is a maximally entangled state shared by system $\mathscr{A}$ and its clone $\mathscr{A}'$.

$$\Phi[X] = d^{\mathscr{A}}\, \mathrm{tr}_{\mathscr{A}'} \left[\Omega_\Phi^{\mathscr{A}\mathscr{A}'} (I_{\rm out}^{\mathscr{A}} \otimes X^{\rm T}) \right], \qquad (2)$$

where $X^{\rm T} = \sum_{i,j} \langle j|X|i\rangle |i'\rangle\langle j'| \in \mathcal{T}(\mathcal{H}_{\rm in}^{\mathscr{A}'})$ and $\mathrm{tr}_{\mathscr{A}'}$ denotes the partial trace operation.

$\{\Phi^{\mathscr{A}}$ is completely positive (CP)$\} \Leftrightarrow \{\Omega_\Phi^{\mathscr{A}\mathscr{A}'} \ge 0\}$.

$\{\Lambda$ is positive$\} \Leftrightarrow \{\Omega_\Lambda^{\mathscr{A}\mathscr{A}'} = \xi_{\rm BP}^{\mathscr{A}|\mathscr{A}'} \in \mathcal{T}(\mathcal{H}_{\rm out}^{\mathscr{A}} \otimes \mathcal{H}_{\rm in}^{\mathscr{A}'})$,

where $\xi_{\rm BP}^{\mathscr{X}|\mathscr{Y}} \in \mathcal{T}(\mathcal{H}^{\mathscr{X}} \otimes \mathcal{H}^{\mathscr{Y}})$ is *block-positive*: $\langle x \otimes y|\xi|x \otimes y\rangle \ge 0$ for all $|x\rangle \in \mathcal{H}^{\mathscr{X}}$, $|y\rangle \in \mathcal{H}^{\mathscr{Y}}$.

Entanglement-breaking maps

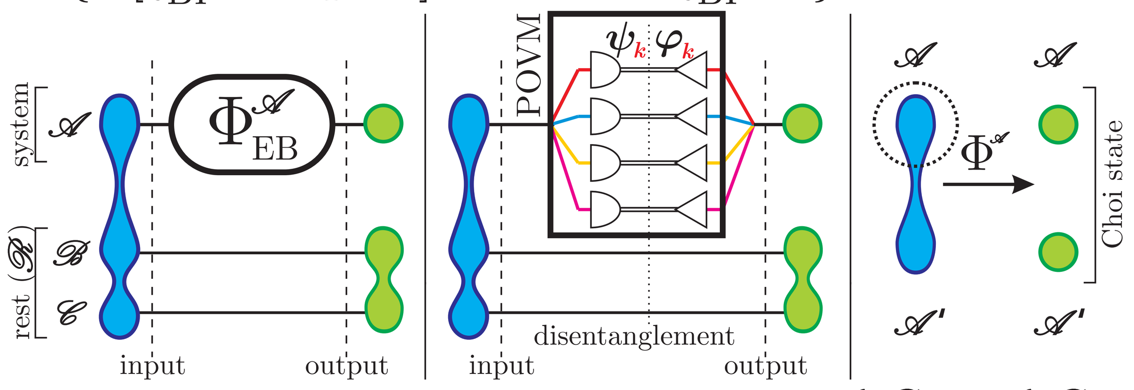
$\{\Phi^{\mathscr{A}}$ is EA$\}$

$\Leftrightarrow \{A_k \propto |\varphi_k\rangle\langle\psi_k|$ with $|\psi_k\rangle, |\varphi_k\rangle \in \mathcal{H}_{\rm in,out}^{\mathscr{A}}\}$

$\Leftrightarrow \{\Phi^{\mathscr{A}}[X] = \sum_k \mathrm{tr}[F_k X]\varrho_k$ for some POVM $\{F_k\}$ and states $\varrho_k \in \mathcal{S}(\mathcal{H}_{\rm out}^{\mathscr{A}})\}$

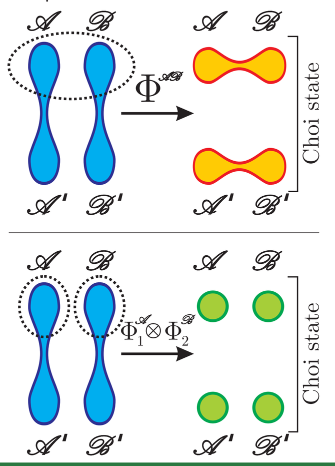
$\Leftrightarrow \{\Omega_\Phi^{\mathscr{A}\mathscr{A}'}$ is separable w.r.t. partition $\mathscr{A}|\mathscr{A}'\}$

$\Leftrightarrow \{\mathrm{tr}[\xi_{\rm BP}^{\mathscr{A}|\mathscr{A}'} \Omega_\Phi^{\mathscr{A}\mathscr{A}'}] \ge 0$ for all $\xi_{\rm BP}^{\mathscr{A}|\mathscr{A}'}\}$

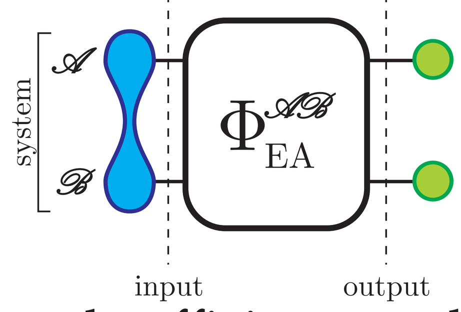


For bipartite maps $\Phi_{\rm EB}^{\mathscr{AB}}$, the Choi state $\Omega_\Phi^{\mathscr{AB}\mathscr{A}'\mathscr{B}'}$ is to be separable w.r.t. partition $\mathscr{AB}|\mathscr{A}'\mathscr{B}'$ but can still be entangled w.r.t. partitions $\mathscr{A}|\mathscr{B}\mathscr{A}'\mathscr{B}'$ and $\mathscr{B}|\mathscr{A}\mathscr{A}'\mathscr{B}'$.

If $\Phi_{\rm local}^{\mathscr{AB}} = \Phi_1^{\mathscr{A}} \otimes \Phi_2^{\mathscr{B}}$, then $\{\Phi_{\rm local}^{\mathscr{AB}}$ is EB$\} \Leftrightarrow \{\Phi_1^{\mathscr{A}}$ is EB and $\Phi_2^{\mathscr{B}}$ is EB$\}$.



Entanglement-annihilating maps

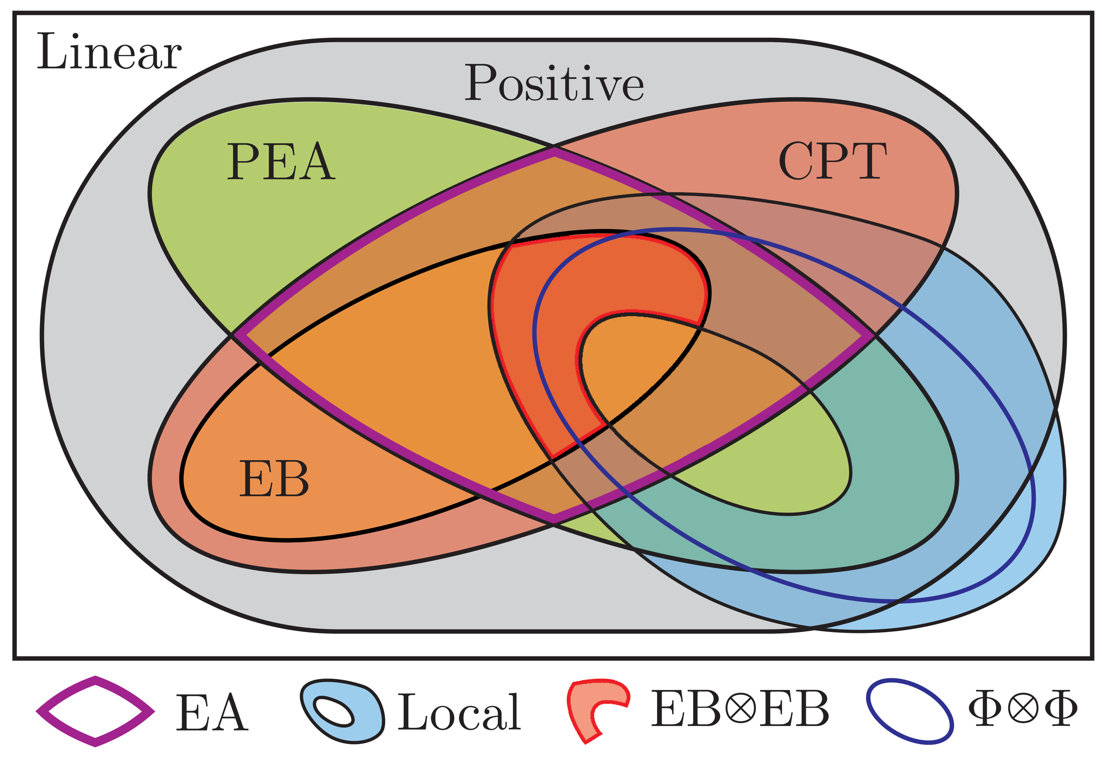


EA channels by definition act on composite systems. For bipartite systems one can use Horodecki criterion to formulate a necessary and sufficient condition for the map to be EA.

Lemma 1. *$\{\Phi^{\mathscr{AB}} : \mathcal{T}(\mathcal{H}_{\rm in}^{\mathscr{AB}}) \to \mathcal{T}(\mathcal{H}_{\rm out}^{\mathscr{AB}})$ is EA$\} \Leftrightarrow \{(\mathrm{Id}^{\mathscr{A}} \otimes \Lambda^{\mathscr{B}}) \circ \Phi^{\mathscr{AB}}$ is a positive map for any positive map $\Lambda^{\mathscr{B}} : \mathcal{T}(\mathcal{H}_{\rm out}^{\mathscr{B}}) \to \mathcal{T}(\mathcal{H}_{\rm out}^{\mathscr{A}})\}$.*

A map $\Phi^{\mathscr{AB}}$ is called *positive entanglement-annihilating* (PEA) if it is positive and $\Phi^{\mathscr{AB}}[\varrho]$ belongs to a cone of separable states $\mathscr{A}|\mathscr{B}$ for *all* $\varrho \in \mathcal{S}(\mathcal{H}_{\rm in}^{\mathscr{AB}})$. EA channels = PEA ∩ CPT.

Structure of linear bipartite maps



Venn diagram of linear bipartite maps $\Phi^{\mathscr{AB}}$. Convex figures correspond to convex sets.

Criteria of entanglement annihilation

Cone of PEA maps is closed under left-composition by partially positive maps $\mathrm{Id}^{\mathscr{A}} \otimes \Lambda^{\mathscr{B}}$ (*left PP-invariant*), which follows from Lemma 1. This fundamental property enables us to characterize PEA maps.

Proposition 1. *The map $\Phi^{\mathscr{AB}}$ is PEA if and only if*

$$\boxed{\mathrm{tr}\left[(\xi_{\rm BP}^{\mathscr{A}|\mathscr{B}} \otimes \varrho^{\mathscr{A}'\mathscr{B}'})\Omega_\Phi^{\mathscr{AB}\mathscr{A}'\mathscr{B}'}\right] \ge 0} \qquad (3)$$

for all block-positive $\xi_{\rm BP}^{\mathscr{A}|\mathscr{B}}$ and $\varrho^{\mathscr{A}'\mathscr{B}'} \in \mathcal{S}(\mathcal{H}^{\mathscr{A}'} \otimes \mathcal{H}^{\mathscr{B}'})$.

Proof. Using Lemma 1, we get $\{\Phi^{\mathscr{AB}}$ is PEA$\} \Leftrightarrow \{(\mathrm{Id}^{\mathscr{A}} \otimes \Lambda^{\mathscr{B}}) \circ \Phi^{\mathscr{AB}}$ is a positive map for any positive map $\Lambda^{\mathscr{B}}\}$, which is equivalent to the block-positivity of matrix $\Omega_{(\mathrm{Id}\otimes\Lambda)\circ\Phi}^{\mathscr{AB}\mathscr{A}'\mathscr{B}'} \equiv (\mathrm{Id}^{\mathscr{A}\mathscr{A}'\mathscr{B}'} \otimes \Lambda^{\mathscr{B}})[\Omega_\Phi^{\mathscr{AB}\mathscr{A}'\mathscr{B}'}]$:

$$0 \le \langle \varphi^{\mathscr{AB}} \otimes \chi^{\mathscr{A}'\mathscr{B}'}|(\mathrm{Id}^{\mathscr{A}\mathscr{A}'\mathscr{B}'} \otimes \Lambda^{\mathscr{B}})[\Omega_\Phi^{\mathscr{AB}\mathscr{A}'\mathscr{B}'}]|\varphi^{\mathscr{AB}} \otimes \chi^{\mathscr{A}'\mathscr{B}'}\rangle$$
$$= \mathrm{tr}\Big[(\mathrm{Id}^{\mathscr{A}} \otimes \Lambda^{\dagger\mathscr{B}})[|\varphi^{\mathscr{AB}}\rangle\langle\varphi^{\mathscr{AB}}|] \otimes |\chi^{\mathscr{A}'\mathscr{B}'}\rangle\langle\chi^{\mathscr{A}'\mathscr{B}'}| \Omega_\Phi^{\mathscr{AB}\mathscr{A}'\mathscr{B}'}\Big],$$

where $\Lambda^\dagger$ is the dual map: $\mathrm{tr}[X\Lambda[Y]] \equiv \mathrm{tr}[\Lambda^\dagger[X]Y]$. Since $\Lambda^\dagger$ is a positive map, the operator $(\mathrm{Id}^{\mathscr{A}} \otimes \Lambda^{\dagger\mathscr{B}})[|\varphi^{\mathscr{AB}}\rangle\langle\varphi^{\mathscr{AB}}|]$ is block-positive (equals $\xi_{\rm BP}^{\mathscr{A}|\mathscr{B}}$). Arbitrariness of $\Lambda, |\varphi\rangle, |\chi\rangle$ results in formula (3). □

The cone of PEA maps is dual to the cone of maps $\Phi^{\mathscr{AB}}[X] = \sum_k \mathrm{tr}[F_k X]\xi_{{\rm BP}\,k}^{\mathscr{A}|\mathscr{B}}$, $F_k \ge 0$.

Corollary 1. *The map $\Phi^{\mathscr{AB}}$ is an EA channel if and only if its Choi matrix $\Omega_\Phi^{\mathscr{AB}\mathscr{A}'\mathscr{B}'}$ satisfies (3), $\Omega_\Phi^{\mathscr{AB}\mathscr{A}'\mathscr{B}'} \ge 0$, and $\mathrm{tr}_{\mathscr{AB}}\Omega_\Phi^{\mathscr{AB}\mathscr{A}'\mathscr{B}'} = (d^{\mathscr{A}} d^{\mathscr{B}})^{-1} I^{\mathscr{A}'\mathscr{B}'}$.*

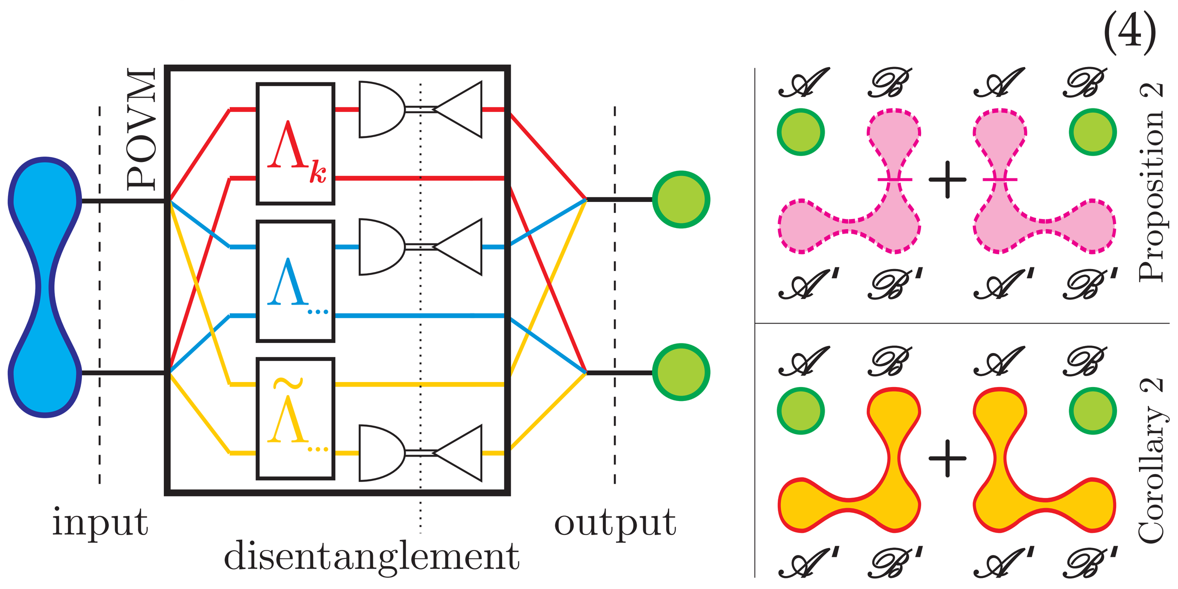
Proof. The tree requirements guarantee that $\Phi \in$ PEA, $\Phi \in$ CP, and Φ is trace-preserving, respectively. □

Proposition 2. *If $\Omega_\Phi^{\mathscr{AB}\mathscr{A}'\mathscr{B}'}$ can be written as a convex sum of operators $\zeta_{\rm BP}^{\mathscr{A}|\mathscr{A}'\mathscr{B}'} \otimes \varrho^{\mathscr{B}}$ and $\varrho^{\mathscr{A}} \otimes \zeta_{\rm BP}^{\mathscr{B}|\mathscr{A}'\mathscr{B}'}$, where $\zeta_{\rm BP}$ is block-positive w.r.t. corresponding cut and ϱ is positive, then the map $\Phi^{\mathscr{AB}\mathscr{A}'\mathscr{B}'}$ is PEA.*

Proof. Substituting $\zeta_{\rm BP}^{\mathscr{A}|\mathscr{A}'\mathscr{B}'} \otimes \varrho^{\mathscr{B}}$ for $\Omega_\Phi^{\mathscr{AB}\mathscr{A}'\mathscr{B}'}$ in (3), we obtain: $\mathrm{tr}_{\mathscr{A}'\mathscr{B}'}[\zeta_{\rm BP}^{\mathscr{A}|\mathscr{A}'\mathscr{B}'} |\chi^{\mathscr{A}'\mathscr{B}'}\rangle\langle\chi^{\mathscr{A}'\mathscr{B}'}|] = \tilde{\varrho}^{\mathscr{A}} \ge 0$ and $\mathrm{tr}_{\mathscr{AB}}[\xi_{\rm BP}^{\mathscr{A}|\mathscr{B}} \tilde{\varrho}^{\mathscr{A}} \otimes \varrho^{\mathscr{B}}] \ge 0$, i.e. (3) holds true. By exchanging $\mathscr{A} \leftrightarrow \mathscr{B}$ it is clear that the operator $\varrho^{\mathscr{A}} \otimes \xi_{\rm BP}^{\mathscr{B}|\mathscr{A}'\mathscr{B}'}$ also satisfies the requirement (3). □

The corresponding map is a sum of concatenations of some positive maps $\Lambda_k^{\mathscr{AB}}$ followed by EB operations $\mathcal{O}_{{\rm EB}_k}[\bullet] = A_k \bullet A_k^\dagger$ acting on one of subsystems:

$$\Phi^{\mathscr{AB}} = \sum_k \{(\mathcal{O}_{{\rm EB}_k}^{\mathscr{A}} \otimes \mathrm{Id}^{\mathscr{B}}) \circ \Lambda_k^{\mathscr{AB}} + (\mathrm{Id}^{\mathscr{A}} \otimes \tilde{\mathcal{O}}_{{\rm EB}_k}^{\mathscr{B}}) \circ \tilde{\Lambda}_k^{\mathscr{AB}}\} \qquad (4)$$



Corollary 2. *If $\mathrm{tr}_{\mathscr{AB}}\Omega_\Phi^{\mathscr{AB}\mathscr{A}'\mathscr{B}'} = (d^{\mathscr{A}} d^{\mathscr{B}})^{-1} I^{\mathscr{A}'\mathscr{B}'}$ and $\Omega_\Phi^{\mathscr{AB}\mathscr{A}'\mathscr{B}'}$ is a convex sum of operators $\varrho^{\mathscr{A}|\mathscr{B}\mathscr{A}'\mathscr{B}'}$ and $\varrho^{\mathscr{B}|\mathscr{A}\mathscr{A}'\mathscr{B}'}$ (separable w.r.t. partitions $\mathscr{A}|\mathscr{B}\mathscr{A}'\mathscr{B}'$ and $\mathscr{B}|\mathscr{A}\mathscr{A}'\mathscr{B}'$, respectively), then $\Phi^{\mathscr{AB}}$ is an EA channel.*

Applicability to local and global depolarizing noises

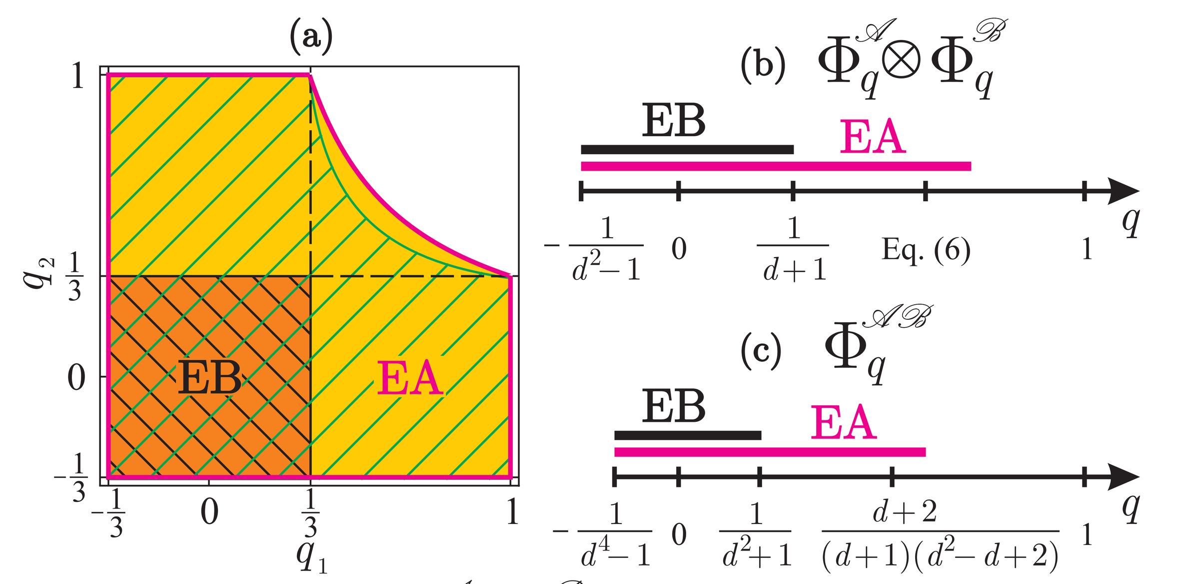
Depolarizing map: $\Phi_q = q\mathrm{Id} + (1-q)\mathrm{Tr}$, where $\mathrm{Tr}[X] = \mathrm{tr}[X]\frac{1}{d}I_d$, $d = \dim\mathcal{H}$. Local noise: $\Phi_{q_1}^{\mathscr{A}} \otimes \Phi_{q_2}^{\mathscr{B}}$. Consider a bipartite system $\mathscr{AB}$ with $d^{\mathscr{A}} = d^{\mathscr{B}} = d$. Resolution (4) holds true ($\Phi_{q_1}^{\mathscr{A}} \otimes \Phi_{q_2}^{\mathscr{B}}$ is EA) if

$$(d^2 - 1)\, q_1 q_2 \le 1 + \tfrac{(d-2)(d+1)}{d+2}(q_1 + q_2). \qquad (5)$$

If $q_{1,2} = q$, then $\Phi_q \otimes \Phi_q$ is surely EA if

$$q \le \frac{d - 2 + d\sqrt{\frac{2d}{d+1}}}{(d-1)(d+2)} \underset{d \gg 1}{\approx} \frac{\sqrt{2}+1}{d}\left(1 - \frac{\sqrt{3}}{\sqrt{2}d}\right). \qquad (6)$$

Global depolarizing channel $\Phi_q^{\mathscr{AB}}$ acting on two d-dimensional subsystems $\mathscr{A}$ and $\mathscr{B}$ simultaneously is EB if $q \le \frac{1}{d^2+1}$ and is EA (disentangles $\mathscr{A}$ from $\mathscr{B}$) if $q \le (d+2)/(d+1)(d^2-d+2)$, when (4) takes place.



(a) Two-qubit channel $\Phi_{q_1}^{\mathscr{A}} \otimes \Phi_{q_2}^{\mathscr{B}}$ [4]. Proposition 2 provides necessary and sufficient condition of EA, Corollary 2 detects EA maps inside the green hatching. EB and EA regions of local (b) and global (c) depolarizing noises acting on two qudits.

References

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