MUBs and SIC-POVMs in view of star-product formalism and tomographic-probability representation of qudits

Sergey N. Filippov

Moscow Institute of Physics and Technology, Russia

http://filippovsn.fizteh.ru/

Content

1. Star-product scheme
2. Mutually unbiased bases
3. Symmetric informationally complete POVM

Symbols of operators acting on $\mathscr{H}_d$

"Dequantization": $f_A(k) = \mathrm{Tr}\left[\hat{A}\hat{U}_k\right]$

"Quantization": $\hat{A} = \sum_k f_A(k)\hat{D}_k$

Simple properties:

$$\mathrm{Tr}\left[\hat{U}_{k'}\hat{D}_k\right] = \delta(k, k'),$$

$$\sum_{k'} f_A(k')\delta(k, k') = f_A(k).$$

★-product scheme

Symbol of the operator $\hat{A}\hat{B}$ is

$$(f_A \star f_B)(k) \equiv f_{AB}(k) = \sum_{k',k''} f_A(k') f_B(k'') K(k', k'', k),$$

where the kernel K is expressed in terms of dequantizer and quantizer operators

$$K(k', k'', k) = \mathrm{Tr}\big[\hat{D}_{k'} \hat{D}_{k''} \hat{U}_k\big].$$

Associativity:

$$f_{ABC}(\mathbf{x}) = (f_A \star f_B \star f_C)(\mathbf{x}) = \big((f_A \star f_B) \star f_C\big)(\mathbf{x}) = \big(f_A \star (f_B \star f_C)\big)(\mathbf{x})$$

$$K^{(3)}(k', k'', k''', k) = \sum_l K(k', k'', l) K(l, k''', k) = \sum_l K(k', l, k) K(k'', k''', l).$$

$\hat{A}\hat{B}\hat{C}\hat{D}$:

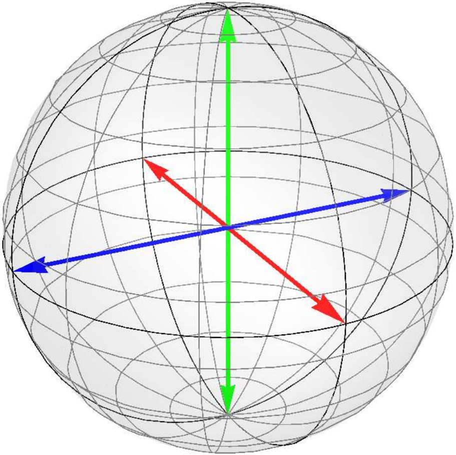
$$
\begin{aligned}
& K^{(4)}(k', k'', k''', k'''', k) \\
& = \sum_{l,m} K(k', k'', l) K(l, k''', m) K(m, k'''', k) \\
& = \sum_{l,m} K(k', k'', l) K(l, m, k) K(k''', k'''', m) \\
& = \sum_{l,m} K(k', l, m) K(k'', k''', l) K(m, k'''', k) \\
& = \sum_{l,m} K(k', l, k) K(k'', k''', m) K(m, k'''', l) \\
& = \sum_{l,m} K(k', l, k) K(k'', m, l) K(k''', k'''', m)
\end{aligned}
$$

Mutually unbiased bases (MUBs)

Full set of MUBs is a collection of $d+1$ bases $\{|a\alpha\rangle\}_{\alpha=0}^{d-1}$
($a = 0, \dots, d$ is the basis number,
$\alpha = 0, \dots, d-1$ refers to one of the states belonging to basis a)
such that

$$|\langle a\alpha|b\beta\rangle|^2 = \frac{1}{d}(1-\delta_{a,b}) + \delta_{a,b}\delta_{\alpha,\beta} \tag{1}$$

$$|\langle a\alpha|b\beta\rangle|^2 = \frac{1}{d} \quad \text{if} \quad a \neq b$$



MUB tomography

"Scanning": $$p_{a\alpha} = \langle a\alpha|\hat{\rho}|a\alpha\rangle = \mathrm{Tr}\left[\hat{\rho}\hat{\Pi}_{a\alpha}\right]$$

$$\sum_{\alpha=0}^{d-1} p_{a\alpha} = 1, \quad \sum_{a=0}^{d}\sum_{\alpha=0}^{d-1} p_{a\alpha} = d+1$$

"Reconstruction":

$$\hat{\rho} = \sum_{b=0}^{d}\sum_{\beta=0}^{d-1} p_{b\beta}\left(\hat{\Pi}_{b\beta} - \frac{1}{d+1}\hat{I}\right)$$

MUB ★-product scheme

$$\hat{U}_{a\alpha} = \hat{\Pi}_{a\alpha}, \quad \hat{D}_{a\alpha} = \hat{\Pi}_{a\alpha} - \frac{1}{d+1}\hat{I}$$

"Dequantization": $$f_A(a,\alpha) = \mathrm{Tr}\left[\hat{A}\hat{\Pi}_{a\alpha}\right]$$

"Quantization": $$\hat{A} = \sum_{a=0}^{d}\sum_{\alpha=0}^{d-1} f_A(a,\alpha)\left(\hat{\Pi}_{a\alpha} - \frac{1}{d+1}\hat{I}\right)$$

Delta function on symbols

$$\begin{aligned}\delta(a,\alpha;b,\beta) &= \mathrm{Tr}\left[\hat{D}_{a\alpha}\hat{U}_{b\beta}\right] = \mathrm{Tr}\left[\hat{\Pi}_{a\alpha}\hat{\Pi}_{b\beta}\right] - \frac{1}{d+1} \\ &= \frac{1}{d(d+1)} + \delta_{a,b}\left(\delta_{\alpha,\beta} - \frac{1}{d}\right)\end{aligned}$$

MUB triple product $T_{a\alpha,b\beta,c\gamma} = \mathrm{Tr}\left[\hat{\Pi}_{a\alpha}\hat{\Pi}_{b\beta}\hat{\Pi}_{c\gamma}\right]$

MUB star-product kernel

$$K(a,\alpha;b,\beta;c,\gamma) = \mathrm{Tr}\left[\hat{D}_{a\alpha}\hat{D}_{b\beta}\hat{U}_{c\gamma}\right]$$
$$= T_{a\alpha,b\beta,c\gamma} + \frac{\delta_{a,c}+\delta_{b,c}}{d(d+1)} - \frac{\delta_{a,c}\delta_{\alpha,\gamma}+\delta_{b,c}\delta_{\beta,\gamma}}{d+1} - \frac{d+2}{d(d+1)^2}$$

4-product

$$\mathrm{Tr}\left[\hat{\Pi}_{a\alpha}\hat{\Pi}_{b\beta}\hat{\Pi}_{k\varkappa}\hat{\Pi}_{l\lambda}\right] = \sum_{c=0}^{d}\sum_{\gamma=0}^{d-1} T_{a\alpha,b\beta,c\gamma}T_{c\gamma,k\varkappa,l\lambda}$$
$$-\left(\frac{1}{d}(1-\delta_{a,b})+\delta_{a,b}\delta_{\alpha,\beta}\right)\left(\frac{1}{d}(1-\delta_{k,l})+\delta_{k,l}\delta_{\varkappa,\lambda}\right)$$

Symmetric informationally complete POVM (SIC-POVM)

SIC-POVM is a set of d^2 effects of the form $E_i = \frac{1}{d}\hat{\Pi}_i$, where rank-1 projectors $\hat{\Pi}_i = |\psi_i\rangle\langle\psi_i|$ satisfy the relation

$$\mathrm{Tr}\left[\hat{\Pi}_i \hat{\Pi}_j\right] = |\langle\psi_i|\psi_j\rangle|^2 = \frac{d\,\delta_{ij} + 1}{d+1}$$

(δ_{ij} is the Kronecker delta symbol)

Problem. To prove or disprove the existence of SIC-POVM in Hilbert space $\mathscr{H}_d$ of an arbitrary dimension d.

Achieved so far results: the existence is proved analytically in dimensions $d = 2, \ldots, 15, 19, 24$ and numerically for $d \leq 67$.

Example. Symmetric set of vectors in $\mathscr{H}_2$:

$$|\psi_1\rangle = \frac{1}{\sqrt{2\sqrt{3}}} \begin{pmatrix} \sqrt{\sqrt{3}+1} \\ \sqrt{\sqrt{3}-1}\ e^{i\pi/4} \end{pmatrix},$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2\sqrt{3}}} \begin{pmatrix} \sqrt{\sqrt{3}-1} \\ \sqrt{\sqrt{3}+1}\ e^{-i\pi/4} \end{pmatrix},$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2\sqrt{3}}} \begin{pmatrix} \sqrt{\sqrt{3}-1} \\ \sqrt{\sqrt{3}+1}\ e^{i3\pi/4} \end{pmatrix},$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2\sqrt{3}}} \begin{pmatrix} \sqrt{\sqrt{3}+1} \\ \sqrt{\sqrt{3}-1}\ e^{-i3\pi/4} \end{pmatrix}.$$

SIC-projectors in $\mathscr{H}_2$:

$$\hat{\Pi}_1 = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3}+1 & 1-i \\ 1+i & \sqrt{3}-1 \end{pmatrix},$$

$$\hat{\Pi}_2 = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3}-1 & 1+i \\ 1-i & \sqrt{3}+1 \end{pmatrix},$$

$$\hat{\Pi}_3 = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3}-1 & -1-i \\ -1+i & \sqrt{3}+1 \end{pmatrix},$$

$$\hat{\Pi}_4 = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3}+1 & -1+i \\ -1-i & \sqrt{3}-1 \end{pmatrix}.$$

Note that $\mathrm{Tr}\left[\hat{\Pi}_i\right] = \mathrm{Tr}\left[\hat{\Pi}_i^2\right] = \mathrm{Tr}\left[\hat{\Pi}_i^3\right] = 1$ for all $i = 1, \ldots, 4$

$\mathrm{Tr}\left[\hat{\Pi}_i\hat{\Pi}_j\right] = 1/3$ if $i \neq j$.

Projectors $\{\hat{\Pi}_i\}_{i=1}^4$ form a basis of operators acting on $\mathscr{H}_2$ (linear space of 2×2 matrices).

Illustration of SIC-projectors:

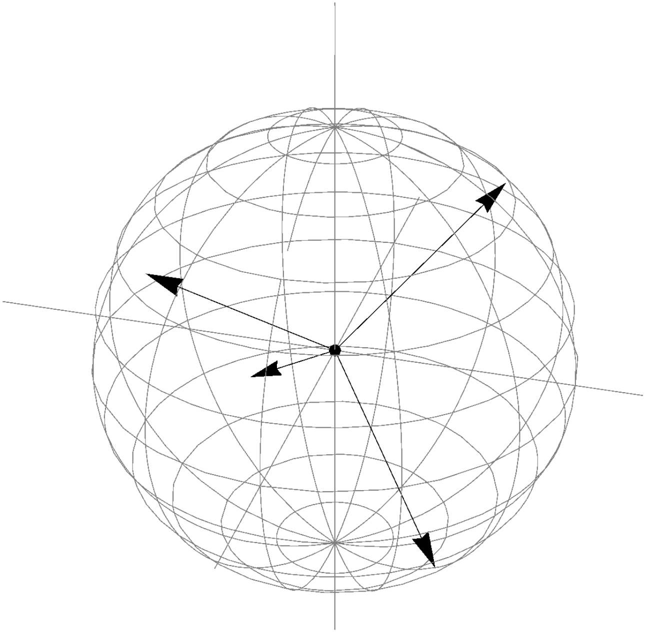
$$\hat{\Pi}_i = \frac{1}{2}\left(\hat{I} + (\hat{\boldsymbol{\sigma}} \cdot \mathbf{n}_i)\right) = \frac{1}{2}\left(\hat{I} + n_{ix}\hat{\sigma}_x + n_{iy}\hat{\sigma}_y + n_{iz}\hat{\sigma}_z\right),$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

unit vectors

$$\mathbf{n}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix},\ \mathbf{n}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & -1 & -1 \end{pmatrix},$$
$$\mathbf{n}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & 1 & -1 \end{pmatrix},\ \mathbf{n}_4 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & -1 & 1 \end{pmatrix}.$$



Motivation is SIC-tomography

For any operator $\hat{\rho}$ acting on $\mathscr{H}_d$ we have

$$p_i = \frac{1}{d}\mathrm{Tr}\big[\hat{\rho}\hat{\Pi}_i\big], \qquad \hat{\rho} = (d+1)\sum_{i=1}^{d^2} p_i\hat{\Pi}_i - \hat{I}. \tag{2}$$

For density operator ($\hat{\rho} \geq 0$, $\mathrm{Tr}\big[\hat{\rho}\big] = 1$) numbers $\{p_i\}_{i=1}^{d^2}$ are probabilities.

SIC ★-product scheme

$$p_i = \frac{1}{d}\mathrm{Tr}\big[\hat{\rho}\hat{\Pi}_i\big], \qquad \hat{\rho} = (d+1)\sum_{i=1}^{d^2} p_i\hat{\Pi}_i - \hat{I}.$$

$$\hat{U}_i = \frac{1}{d}\hat{\Pi}_i \qquad \text{and} \qquad \hat{D}_i = (d+1)\hat{\Pi}_i - \hat{I}, \qquad i = 1,\ldots,d^2$$

Notation. $T_{ijk} = \mathrm{Tr}\big[\hat{\Pi}_i\hat{\Pi}_j\hat{\Pi}_k\big]$ is triple product.

$$\mathfrak{D}_{ij} = \mathrm{Tr}\big[\hat{U}_i\hat{D}_j\big] = \delta_{ij},$$

$$K_{ijk} = \frac{1}{d}\left[(d+1)^2 T_{ijk} - d(\delta_{ik}+\delta_{jk}) - 1\right].$$

$$\sum_{m=1}^{d^2}\big(T_{ijm}T_{mkl} - T_{iml}T_{jkm}\big)$$

$$= \frac{d}{(d+1)^3}\Big[(d\delta_{ij}+1)(d\delta_{kl}+1) - (d\delta_{jk}+1)(d\delta_{il}+1)\Big],$$

4-product

$$\mathrm{Tr}\left[\hat{\Pi}_i\hat{\Pi}_j\hat{\Pi}_k\hat{\Pi}_l\right] = \frac{d+1}{d}\sum_{m=1}^{d^2} T_{ijm}T_{mkl} - \frac{(d\delta_{ij}+1)(d\delta_{kl}+1)}{(d+1)^2}.$$

5-product

$$\mathrm{Tr}\left[\hat{\Pi}_i\hat{\Pi}_j\hat{\Pi}_k\hat{\Pi}_l\hat{\Pi}_m\right] = \frac{(d+1)^2}{d^2}\sum_{n,p=1}^{d^2} T_{ijn}T_{nkp}T_{plm}$$
$$- T_{ijk}\frac{d\delta_{lm}+1}{d+1} - \frac{(d\delta_{ij}+1)(d\delta_{lm}+1)}{(d+1)^2} - \frac{d\delta_{ij}+1}{d+1}T_{klm}.$$

Qubits: $d = 2$

$$T_{ijk} = \frac{1}{4}\Big\{1 + (\mathbf{r}_i \cdot \mathbf{r}_j) + (\mathbf{r}_j \cdot \mathbf{r}_k) + (\mathbf{r}_k \cdot \mathbf{r}_i) + \mathrm{i}(\mathbf{r}_i \cdot [\mathbf{r}_j \times \mathbf{r}_k])\Big\},$$

In our case $(\mathbf{r}_i \cdot \mathbf{r}_j) = (4\delta_{ij} - 1)/3$ and $(\mathbf{r}_i \cdot [\mathbf{r}_j \times \mathbf{r}_k]) = -4\varepsilon_{ijk}/3\sqrt{3}$, where ε_{ijk} is antisymmetric with respect to permutations and $\varepsilon_{123} = \varepsilon_{134} = \varepsilon_{142} = \varepsilon_{432} = 1$.

$$T_{ijk} = \frac{1}{3}\Big\{\delta_{ij} + \delta_{jk} + \delta_{ki} - \frac{\mathrm{i}}{\sqrt{3}}\varepsilon_{ijk}\Big\},$$

$$K_{ijk} = \frac{1}{2}\Big\{3\delta_{ij} - \mathrm{i}\sqrt{3}\varepsilon_{ijk} - 1\Big\}.$$

Conclusion. The full set of MUBs and SIC-POVM (if they do exist) must obey general rules of the star product.

Further information:

- Mutually unbiased bases: tomography of spin states and star-product scheme // Phys. Scr. T143, 014010 (2011)
- SIC-POVM and probability representation of quantum mechanics // J. Rus. Laser Res. 31, 211 (2010), arXiv: 1005.4091 [quant-ph]
- Different bases of operators acting on Hilbert spaces // J. Rus. Laser Res. 32, 56 (2011), arXiv: 1012.6045 [quant-ph]

Thank you for listening!

http://filippovsn.fizteh.ru/

Bon appétit!