Lectures: 1., 2. } Quantum tomography (measurements)
3. Statistical comparison of quantum states
4., 5. } Dynamics of entanglement

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Episode 1

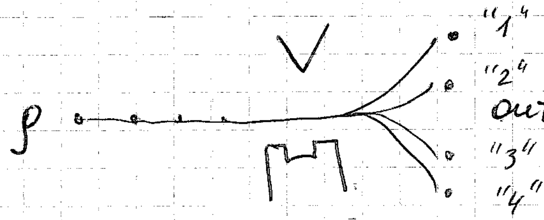
Skipped mathematics:

Def. Hilbert space is a complete inner-product space (i.e., every absolutely convergent series is convergent)

Def.: $\mathcal{H}$ is separable if it has a countable orthonormal basis

Def.: Set of quantum states $S(\mathcal{H}) = \{\rho \mid \underset{\rho=\rho^+}{\rho \geq 0},\ \mathrm{Tr}\,\rho = 1\}$

Example of measurement: Stern-Gerlach apparatus



outcomes, denote outcomes "m"

Probability to obtain outcome "m" equals $p_m = tr[\rho E_m]$ (the Born rule),

where $E_m^+ = E_m$, $E_m \geq 0$, $\sum\limits_m E_m = I$

<u>Not rigorous Def.</u>: $\{E_m\}$ is a POVM (positive operator-~~valued~~ valued measure)

If E satisfies $E^+ = E$, $E \geq 0$, $E \leq I$, let us call E <u>effect</u>.

Let $\mathcal{E}(\mathcal{H})$ be a set of all possible effects

Let Ω be a set of all outcomes and $\mathcal{F}$ be a σ-algebra on Ω.

<u>Rigorous Def</u>: POVM is a mapping $A: \mathcal{F} \to \mathcal{E}(\mathcal{H})$ satisfying

$$A(\varnothing) = O \text{ (null operator)} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & & & & \ddots \end{pmatrix}$$

$$A(\Omega) = I \text{ (identity operator)} \quad \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & & & \ddots \end{pmatrix}$$

$$A\left(\bigcup_i X_i\right) = \sum_i A(X_i)$$

(in the weak sense) for any sequence $\{X_i\}$ of disjoint sets in $\mathcal{F}$.

"Observable" is a synonym of "POVM"

When do observables $\mathcal{A}, \mathcal{B}, \dots$ define ~~the~~ any state?

Def.: A collection $\{\mathcal{A}, \mathcal{B}, \dots\}$ of observables is **informationally complete** (info-complete) if for every $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{H})$

$$\left\{ \begin{array}{l} \text{probability} \\ \text{distributions} \\ \text{of outcomes} \end{array} \begin{array}{l} \Phi_{\mathcal{A}}(\rho_1) = \Phi_{\mathcal{A}}(\rho_2) \\ \Phi_{\mathcal{B}}(\rho_1) = \Phi_{\mathcal{B}}(\rho_2) \\ \dots \end{array} \right\} \Longrightarrow \rho_1 = \rho_2 .$$

In case $\dim \mathcal{H} < \infty$ any finite collection of observables can be considered as a single observable.

Proposition. Suppose $\dim \mathcal{H} < \infty$. An observable $\mathcal{A}$ is informationally complete if and only if every selfadjoint operator can be written as a real linear combination of elements belonging to the range of $\mathcal{A}$.

Informally: POVM $\{E_m\}$ "contains" a basis of operators acting on $\mathcal{H}$

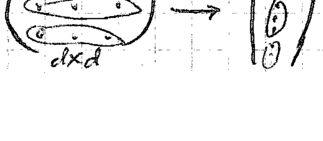
Suppose POVM $\{E_m\}$ is info-complete. How to find ρ if all $\{p_m\}$ are known?

Reconstruction procedure

Two cases

Number of POVM effects
$N > (\dim \mathcal{H})^2 \equiv d^2$

One of possible solutions is to consider operators $\{E_m\}$ as vectors $|E_m\rangle \in \mathcal{H}_{d^2}$



Number of POVM effects
$N = (\dim \mathcal{H})^2 \equiv d^2$

So-called "minimal POVM"

Construct $d^2 \times d^2$ Gram matrix

$$G_{mm'} = \mathrm{tr}[E_m E_{m'}]$$

Invert it and define $R_m = \sum_{m'} G^{-1}_{mm'} E_{m'}$

Then $p_m = \langle E_m | \rho \rangle$, where $|\rho\rangle \in \mathcal{H}_{d^2}$ is a vector corresponding to the density matrix.

Then state ρ is given by formula

$$\hat{\rho} = \sum_{m=1}^{d^2} p_m \hat{R}_m$$

Analoguesly, "quantizer" operators $\hat{R}_m$ (used in $\hat{\rho} = \sum_{m=1}^{N} p_m \hat{R}_m$) can be represented by vectors $|R_m\rangle$

$\sum_m |R_m\rangle\langle E_m|$ is an operator acting on $\mathcal{H}_{d^2}$. (superoperator)

$$\left(\sum_m |R_m\rangle\langle E_m|\right)|\rho\rangle = \sum_m |R_m\rangle \underbrace{\langle E_m|\rho\rangle}_{p_m} = |\rho\rangle \text{ for all } |\rho\rangle$$

$$\Rightarrow \sum_m |R_m\rangle\langle E_m| = I_{d^2} \quad \text{– (identity operator, super-identity)} \quad (*)$$

If we join all the vectors $|E_m\rangle$ into a single matrix

$\underset{d^2\times N}{M_D} = \left(|E_1\rangle, |E_2\rangle, \dots, |E_N\rangle\right)$, it will be $d^2\times N$ matrix

Similarly, $\underset{d^2\times N}{M_Q} = \left(|R_1\rangle, |R_2\rangle, \dots, |R_N\rangle\right)$

$(*) \Rightarrow \quad M_Q \cdot M_D^{+} = I_{d^2}$

$\{$Info-completeness of $\{E_m\}\} \Leftrightarrow \text{rank } M_D = d^2$

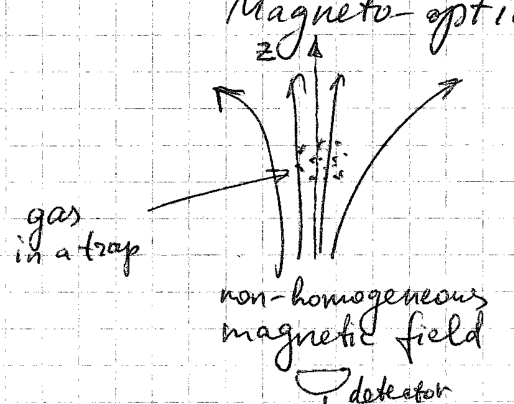
$$M_Q = (M_D \cdot M_D^{+})^{-1} \cdot M_D$$

vectors $|R_m\rangle$ are known (one of many possible solutions)

$\{R_m\}$

Example. Modification of Stern-Gerlach experiment

Magneto-optical trap



We are interested in the "spin" state of ensemble of atoms.

For our purposes "spin" state will be a density matrix associated with the total angular moment F ($F=4$ for Klose's experiment)

For simplicity let us denote F as j. Then states $|jm\rangle$, $m=-j,\dots,j$ form a basis. Hilbert space $\mathcal{H}_d$ has dimension $d=2j+1$.

Upon Applying non-homogeneous magnetic field (Fig.), atoms will be accelerated according to their spin projection m and arrive to the detector at different times. One can measure

population of each level

$$\langle jm|\rho|jm\rangle = tr[\rho \underbrace{|jm\rangle\langle jm|}_{E_m^z}]$$

POVM'-effect

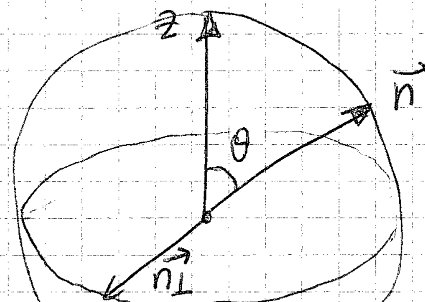
$$(E_m^z)^+ = E_m^z$$

$$E_m^z \geq 0$$

$$\sum_m E_m^z = I$$

This measurement has $(2j+1)$ outcomes and is not info-complete. One needs to repeat the experiment with different direction of magnetic field, $\vec{n}$, say. And so on. How many directions are sufficient? Necessary?

To answer this question consider unitary operator $U(\vec{n}) = e^{-i(\vec{n}_\perp \cdot \vec{j})}$



where $\vec{n}_\perp = (-\sin\varphi, \cos\varphi, 0)$ and $\vec{n} = (\cos\varphi\sin\theta_0, \sin\varphi\sin\theta_0, \cos$

Then $\langle jm|U(\vec{n})\rho U(\vec{n})^+|jm\rangle$ is the probabilit to obtain spin projection m ~~when~~ onto direction $\vec{n}$. If we fix (N_d) directions, the

$$\boxed{E_m^k = \frac{1}{N_d} U(\vec{n}_k)^+ |jm\rangle\langle jm| U(\vec{n}_k)}$$ form a single POVM

What is N_d? According to previous discussion, $\{E_m^k\}$ should contain a basis of operators acting on $\mathcal{H}_{(2j+1)}$, so the number of linear independent elements in $\{E_m^k\}$ should be equal to $(2j+1)^2$.

$|jm\rangle\langle jm|$ is a linear combination of $I, \hat{J}_z, \hat{J}_z^2, \ldots, \hat{J}_z^{2j}$.
(This fact can be seen by the relation $|jm\rangle\langle jm| = \sum_{L=0}^{2j} S_L(m) S_L(\hat{J}_z)$, where $S_L(\ )$ is a discrete Chebyshev polynomial of variable)

$U(\vec{n}_k)^+ |jm\rangle\langle jm| U(\vec{n}_k)$ is a linear combination of $I, (\vec{j}\cdot\vec{n}_k), (\vec{j}\cdot\vec{n}_k)^2, \ldots, (\vec{j}\cdot\vec{n}_k)^{2j}$.

- There is a single identity operator I
  3 linear independent operators $(\vec{j}\cdot\vec{n}_k)$
  5 linear independent operators $(\vec{j}\cdot\vec{n}_k)^2$
  $\vdots$
  $2\cdot 2j+1$ linear independent operators $(\vec{j}\cdot\vec{n}_k)^{2j}$

(This can also be seen by $tr[S_L(\vec{j}\cdot\vec{n}_k) S_{L'}(\vec{j}\cdot\vec{n}_{k'})] = \delta_{LL'} P_L(\vec{n}_k\cdot\vec{n}_{k'})$ Legendre polynomials)

To conclude, the minimal sufficient number of directions $\boxed{N_d = 2\cdot 2j+1 = 4j+1}$. For $j=\frac{1}{2}$ we obviously have $N_d=3$.

- Exercise: Symmetrical info-complete POVM (SIC-POVM)

$E_m = \frac{1}{d}|\psi_m\rangle\langle\psi_m|$, $d = \dim \mathcal{H}_d$, $\sum_{m=1}^{d^2} E_m = I$, such that

$tr[E_m E_{m'}] = \frac{d\delta_{mm'}+1}{d^2(d+1)}$ for all $m, m' = 1, \ldots, d^2$.

FIND THE RECONSTRUCTION FORMULA!

Hint: Invert the Gram matrix $G_{mm'} = tr[E_m E_{m'}]$.

Answer: $\hat{\rho} = \sum_m p_m \hat{R}_m$,

$$\hat{R}_m = d(d+1)\hat{E}_m - \hat{I}$$

$$\rho = \sum_m p_m (d(d+1)E_m - I) = d(d+1)\left(\sum_m p_m E_m\right) - I$$

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