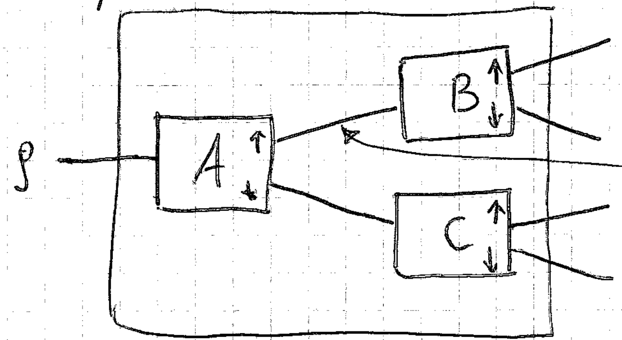
Episode 2.

Sequential measurements



altogether is a 4-outcome POVM.

How to find this POVM? We need to know "conditional" states associated with different outcomes of POVM "A".

"Unphysical" states (also called 'instruments'):

$$\rho_{\uparrow}^{A} = K_{\uparrow}^{A} \rho (K_{\uparrow}^{A})^{+}, \text{ where } K_{\updownarrow}^{A} \text{ — is Kraus operator,}$$

defining also the probability $P_{\uparrow}^{A} = tr[\rho (K_{\uparrow}^{A})^{+} K_{\uparrow}^{A}], E_{\uparrow}^{A} = (K_{\uparrow}^{A})^{+} K_{\uparrow}^{A}$

The probability of outcome $B^{\uparrow}$ is $P_{\uparrow}^{B} = tr[\rho_{\uparrow}^{A} E_{\uparrow}^{B}]$, where $E_{\uparrow}^{B}$ is a POVM-effect (of B-measurement only)

$$P_{\uparrow}^{B} = tr[K_{\uparrow}^{A} \rho (K_{\uparrow}^{A})^{+} E_{\uparrow}^{B}] = tr[\rho \underbrace{(K_{\uparrow}^{A})^{+} E_{\uparrow}^{B} K_{\uparrow}^{A}}_{E_{\uparrow\uparrow}^{AB}}]$$

$E_{\uparrow\uparrow}^{AB}$ – POVM-effect of sequential measurement.

Example: A is "fuzzy" observable such that

$$K_{\uparrow}^{A} = \frac{1}{\sqrt{2\sqrt{3}}} \begin{pmatrix} \sqrt{\sqrt{3}+1} & 0 \\ 0 & \sqrt{\sqrt{3}-1}\, e^{i\frac{\pi}{4}} \end{pmatrix}; \quad K_{\downarrow}^{A} = \frac{1}{\sqrt{2\sqrt{3}}} \begin{pmatrix} \sqrt{\sqrt{3}-1}\, e^{i\frac{\pi}{4}} & 0 \\ 0 & \sqrt{\sqrt{3}+1} \end{pmatrix}$$

B and C are 'sharp' observables with POVM effects

$$E_{\updownarrow}^{B} = \frac{1}{2}(I \pm \sigma_x); \quad E_{\updownarrow}^{C} = \frac{1}{2}(I \pm \sigma_y).$$

Then final 4-outcome POVM will be $E_{\uparrow\uparrow}^{AB}, E_{\uparrow\downarrow}^{AB}, E_{\downarrow\uparrow}^{AC}, E_{\downarrow\downarrow}^{AC}$

All these effects are of the form $\frac{1}{2}(I + \vec{n}_k \cdot \vec{\sigma})$, where

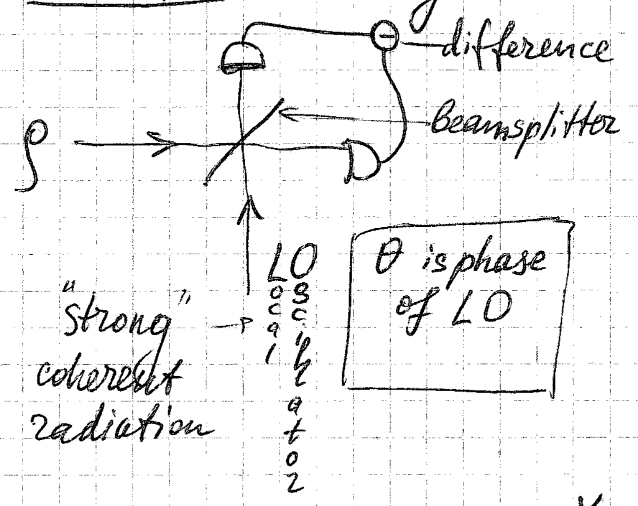
$$\vec{n}_k = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}; \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\-1\end{pmatrix}; \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\-1\end{pmatrix}; \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\-1\\1\end{pmatrix}.$$

We have just constructed SIC-POVM for qubits.

Measurements with "continuous" outcomes

Continuous-variable quantum states (CV-states):

Example: homodyne detector



For fixed θ:

$$\hat{X}_\theta = \frac{1}{\sqrt{2}}\left(\hat{a} e^{-i\theta} + \hat{a}^{\dagger} e^{i\theta}\right),$$

where $\hat{a}, \hat{a}^{\dagger}$ are photon annihilation and creation operators

X_θ is measured (difference photocurrent)

$X_\theta \in (-\infty;+\infty)$, $\hat{X}_\theta |X_\theta\rangle = X|X_\theta\rangle$

POVM-effects: $|X_\theta\rangle\langle X_\theta| = \delta(X\cdot\mathbb{1} - \hat{q}\cdot\cos\theta - \hat{p}\sin\theta)$,

$$\hat{q} = \frac{\hat{a}+\hat{a}^{\dagger}}{\sqrt{2}}; \quad \hat{p} = \frac{\hat{a}-\hat{a}^{\dagger}}{\sqrt{2}i}$$

Measurable probability distribution $w(X,\theta) = \mathrm{Tr}\left[\hat{\rho}\,\delta(X - \hat{q}\cos\theta - \hat{p}\sin\theta)\right]$

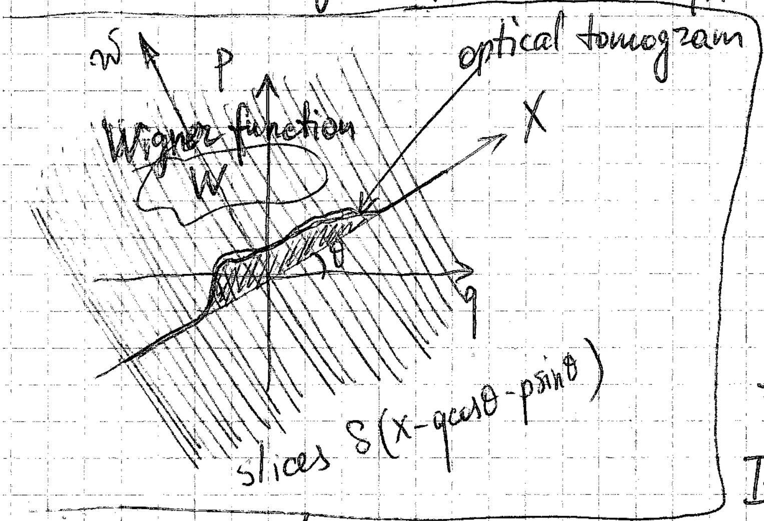
optical tomogram

$w(X,\theta) = \langle X_\theta|\rho|X_\theta\rangle \geq 0$ true probability distribution

Reconstruction formula:

$$\hat{\rho} = \frac{1}{(2\pi)^2}\int w(X,\theta)\, e^{ir(X - \hat{q}\cos\theta - \hat{p}\sin\theta)}\, r\,dr\, dX\, d\theta$$

Proof: Wigner function $W(q,p) = \frac{1}{2\pi}\int \langle q+\frac{y}{2}|\rho|q-\frac{y}{2}\rangle e^{-ipy}\,dy$ is a Weyl-Wigner symbol of operator $\hat{\rho}$.



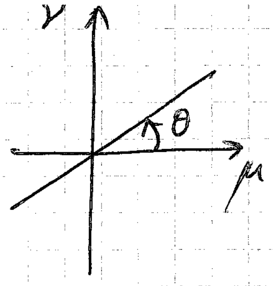
$$w_{sympl}(X,\mu,\nu) = \mathrm{Tr}\left[\hat{\rho}\,\delta(X - \mu\hat{q} - \nu\hat{p})\right] =$$

$$= \int W(q,p)\,\delta(X - \mu q - \nu p)\,dq\,dp =$$

$$= \int W(q,p)\frac{1}{2\pi} e^{ik(X-\mu q - \nu p)}\,dk\,dq\,dp$$

It is easy to check that

$$W(q,p) = \frac{1}{(2\pi)^2}\int w_{sympl}(X,\mu,\nu)\, e^{i(X-\mu q-\nu p)}\,dX\,d\mu\,d\nu$$



$$d\mu\, d\nu = r\,dr\,d\theta$$

$$\mu = r\cos\theta$$

$$\nu = r\sin\theta$$

$$\cancel{w_{sympl}(X, \cos\theta, \sin\theta)}$$

$$w_{sympl}(X, r\cos\theta, r\sin\theta) = \frac{1}{r}\, w_{sympl}(X/r, \cos\theta, \sin\theta) \equiv \frac{1}{r}\, w_{opt}\Big(\overset{\tilde X}{\overset{\parallel}{\tfrac{X}{r}}}, \theta\Big)$$

So, $$W(q,p) = \frac{1}{(2\pi)^2}\int w_{opt}(\tilde X, \theta)\, e^{ir(\tilde X - q\cos\theta - p\sin\theta)}\, r\,dr\, d\tilde X\, d\theta$$

$\left.\begin{matrix} q\to\hat q, \\ p\to\hat p \end{matrix}\right\}$ concludes the proof.

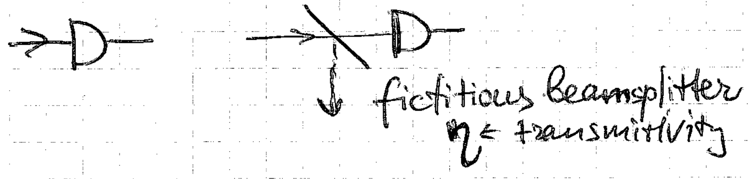
Exercise: Calculate $w_{opt}(X,\theta)$ for coherent state $|\alpha\rangle$ and single photon-added coherent state $\frac{\hat a^\dagger|\alpha\rangle}{\sqrt{1+|\alpha|^2}}$ (SPACS)

Answer: $$w_{|\alpha\rangle}(x,\theta) = \frac{1}{\sqrt{\pi}}\exp\left\{-\left[X - \sqrt{2}\,(\mathrm{Re}\,\alpha\cos\theta + \mathrm{Im}\,\alpha\sin\theta)\right]^2\right\}$$

$$w_{\hat a^\dagger|\alpha\rangle}(x,\theta) = \frac{1}{\sqrt{\pi}\,(1+|\alpha|^2)}\left\{2\left(X - \frac{1}{\sqrt2}(\mathrm{Re}\,\alpha\cos\theta + \mathrm{Im}\,\alpha\sin\theta\right)^2 + \left(\mathrm{Re}\,\alpha\sin\theta - \mathrm{Im}\,\alpha\cdot\cos\theta\right)^2\right\} \times$$

$$\times \exp\left\{-\left[X - \sqrt{2}\,(\mathrm{Re}\,\alpha\cos\theta + \mathrm{Im}\,\alpha\cdot\sin\theta)\right]^2\right\}.$$

Detection imperfection:



$$W(q,p) \to W^{detectable}(q,p) = \frac{1}{\pi(1-\eta)}\int dq'dp'\, W(q',p') \times \exp\left[-\frac{(q-\sqrt{\eta}\,q')^2 + (p-\sqrt{\eta}\,p')^2}{1-\eta}\right]$$

"Practice": Calculate purity
$$\mathrm{tr}[(\hat\rho^{det})^2] = 2\pi\iint [W^{det}(q,p)]^2\, dq\, dp$$
for coherent state and SPACS.

Answer: $|\alpha\rangle$ is transformed into $|\sqrt{\eta}\,\alpha\rangle$, so purity $=1$

$\frac{\hat a^\dagger|\alpha\rangle}{\sqrt{1+|\alpha|^2}} \to$ mixed state, purity $= 1 - \frac{2\eta(1-\eta)}{(1+|\alpha|^2)^2}$

BIBLIOGRAPHY: arXiv: 1203.2974 [quant-ph] and references therein.