Episode 5

Def. A channel $\mathcal{E}$ is called <u>entanglement-breaking</u> (EB) if, for all $\mathcal{H}_{other}$, the state $\omega' = \mathcal{E}\otimes \mathcal{I}_{other}(\omega)$ is separable for all $\omega \in S(\mathcal{H}\otimes\mathcal{H}_{other})$

Horodecki, Shor, and Ruskai formulated the following proposition:

<u>Proposition.</u> A channel $\mathcal{E}$ is entanglement-breaking if and only if $\mathcal{E}$ is of the form $\mathcal{E}(\rho) = \sum_j \xi_j \operatorname{tr}[\rho F_j]$, where the ξ_j are states and the positive operators F_j determines a discrete POVM, i.e. $\sum_j F_j = I$.

<u>Proof.</u> ① $\mathcal{E}$ is EB $\Rightarrow$ $\omega_{\mathcal{E}} = (\mathcal{E}\otimes\mathcal{I})[P_+]$ is separable $\Rightarrow$

$$\Rightarrow \omega_{\mathcal{E}} = \sum_n p_n |\psi_n\rangle\langle\psi_n| \otimes |\varphi_n\rangle\langle\varphi_n|$$

Consider $\mathcal{E}'(\rho) = \sum_n \xi_n \operatorname{tr}[\rho F_n]$ with $\xi_n = |\psi_n\rangle\langle\psi_n|$, $F_n = \underset{\dim\mathcal{H}}{d}\cdot p_n|\varphi_n\rangle\langle\varphi_n|$

Note $\operatorname{tr}_1[\omega_{\mathcal{E}}] = \frac{1}{d} I$, but on the other hand

$\operatorname{tr}_1[\omega_{\mathcal{E}}] = \sum_n p_n |\varphi_n\rangle\langle\varphi_n|$

It follows that $\sum_n F_n = I$, $F_n \geq 0$ $\Rightarrow$ $\{F_n\}$ is a POVM

$$\omega_{\mathcal{E}'} = (\mathcal{E}'\otimes\mathcal{I})[P_+] = \sum_n (\xi_n \otimes I) \operatorname{tr}_1[(F_n\otimes I)P_+] =$$

$$= \underbrace{\sum_n \xi_n \otimes \operatorname{tr}_1[(F_n\otimes I)P_+] = \sum_n p_n |\psi_n\rangle\langle\psi_n| \otimes |\varphi_n\rangle\langle\varphi_n|}_{=\omega_{\mathcal{E}}}$$

Choi-Jamiołkowski isomorphism $\Rightarrow$ $\mathcal{E} = \mathcal{E}'$

② Let $\mathcal{E}(\rho) = \sum_j \xi_j \operatorname{tr}[\rho F_j]$, then $\mathcal{E}$ is EB:

$$(\mathcal{E}\otimes\mathcal{I}_{other})[\Omega] = \sum_j (\xi_j \otimes \mathcal{I}_{other})\left(I\otimes \operatorname{tr}_1\left[(F_j\otimes I_{other})\Omega\right]\right) =$$

$$= \sum_j q_j \xi_j \otimes Q_j,$$

$q_j = \operatorname{tr}[(F_j \otimes I_{other})\Omega]$, $Q_j = \frac{1}{q_j}\cdot \operatorname{tr}_1[(F_j\otimes I_{other})\Omega]$ — separable

This concludes the proof.

Note: to check EB property it is enough to consider max. ent. state P_+

Example: qubits, unital channels $\mathcal{E} = \begin{pmatrix} 1 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \end{pmatrix}$

(matrix representation)

Choi matrix:

$$\frac{1}{2}\begin{pmatrix} 1+\lambda_3 & & & \lambda_1+\lambda_2 \\ & 1-\lambda_3 & \lambda_1-\lambda_2 & \\ & \lambda_1-\lambda_2 & 1-\lambda_3 & \\ \lambda_1+\lambda_2 & & & 1+\lambda_3 \end{pmatrix}$$

should be separable if $\mathcal{E}$ is EB

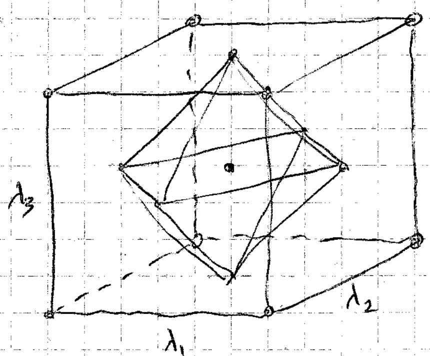
Apply PPT criterion: $\begin{pmatrix} 1+\lambda_3 & & & \lambda_1-\lambda_2 \\ & 1-\lambda_3 & \lambda_1+\lambda_2 & \\ & \lambda_1+\lambda_2 & 1-\lambda_3 & \\ \lambda_1-\lambda_2 & & & 1+\lambda_3 \end{pmatrix} \geqslant 0$

$$\Rightarrow \begin{cases} (1+\lambda_3)^2 - (\lambda_1-\lambda_2)^2 \geqslant 0, \\ (1-\lambda_3)^2 - (\lambda_1+\lambda_2)^2 \geqslant 0. \end{cases} \Leftrightarrow \begin{cases} (1+\lambda_3-\lambda_1+\lambda_2)(1+\lambda_3+\lambda_1-\lambda_2) \geqslant 0, \\ (1-\lambda_3-\lambda_1-\lambda_2)(1-\lambda_3+\lambda_1+\lambda_2) \geqslant 0 \end{cases}$$

$$\Updownarrow$$

$$|1+\lambda_3| \geqslant |\lambda_1-\lambda_2|$$
$$|1-\lambda_3| \geqslant |\lambda_1+\lambda_2|$$

inverTion of tetrahedron considered earlier



this octahedron is the intersection of tetrahedron and inverted tetrahedron

Points $(\lambda_1, \lambda_2, \lambda_3)$ define unital EB qubit channels.

Def. Local two-qubit channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ is entanglement-annihilating (EA) if $(\mathcal{E}_1 \otimes \mathcal{E}_2)[\varrho_{in}]$ is separable for all input states ϱ_{in}.

Def. If channel $\mathcal{E} \otimes \mathcal{E}$ is EA, we will refer to $\mathcal{E}$ as a 2-locally EA (2-LEA) channel.

Properties of EA channels:

1°. Set of all (local and non-local) EA channels is convex

2°. Channel is EA iff it destroys entanglement of all pure input states

3°. If G_{12} is EA (local or non-local), then $G_{12}\cdot F_{12}$ is EA for all two-qubit channels F_{12}

4°. Channels $\mathcal{E}_1 \otimes \mathcal{E}_2$ and $\mathcal{E}_2 \otimes \mathcal{E}_1$ exhibit the same EA behaviour

5°. If $\mathcal{E}_1$ or $\mathcal{E}_2$ is EB, then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA.

6°. Channel $\mathcal{E} \otimes \mathcal{I}$ is EA iff $\mathcal{E}$ is EB

7°. If $\mathcal{E}$ is 2-LEA and $\mathcal{F}$ is EB, then the convex combination $\mu\mathcal{E} + (1-\mu)\mathcal{F}$ is 2-LEA, $\mu \in [0,1]$

Note: $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EB iff $\mathcal{E}_1$ is EB and $\mathcal{E}_2$ is EB

Consequently, if $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EB, then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA by 5°.

Example: depolarizing channels:

$$\mathcal{E}_j [X] = q_j X + (1-q_j)\, \mathrm{tr}[X] \cdot \tfrac{1}{2} I,$$

$$q_j \in [-\tfrac{1}{3}; 1] \quad \text{(to guarantee complete-positivity)}$$

When is $\mathcal{E}_1 \otimes \mathcal{E}_2$ EA?

Use property 2°, $\omega = |\psi\rangle\langle\psi|$

Schmidt decomposition $|\psi\rangle = \sqrt{p}\,|\varphi \otimes \chi\rangle + \sqrt{p_\perp}\,|\varphi_\perp \otimes \chi_\perp\rangle$,

$$p + p_\perp = 1.$$

$$\omega_{out} = (\mathcal{E}_1 \otimes \mathcal{E}_2)[\omega] = q_1 q_2\, \omega + \tfrac{1}{2}(1-q_1) q_2\, I \otimes \omega_2 +$$
$$+ \tfrac{1}{2} q_1 (1-q_2)\, \omega_1 \otimes I + \tfrac{1}{4}(1-q_1)(1-q_2)\, I \otimes I$$

$\omega_1 = p|\varphi\rangle\langle\varphi| + p_\perp |\varphi_\perp\rangle\langle\varphi_\perp|$; $\omega_2 = p|\chi\rangle\langle\chi| + p_\perp |\chi_\perp\rangle\langle\chi_\perp|$

PPT criterion $\Longrightarrow$ $\omega_{out}^{\Gamma} \geq 0 \Rightarrow A^2 - B^2 - C^2 \geq 0$, $A = 1 - q_1 q_2$, $B = (2p-1)(q_1 - q_2)$, $C = 4 q_1 q_2 \sqrt{p p_\perp}$

$$(1 + q_1 q_2)(1 - 3 q_1 q_2) + 4\left(p - \tfrac{1}{2}\right)^2 \left[4 q_1^2 q_2^2 - (q_1 - q_2)^2\right] \geq 0$$

If $2|q_1 q_2| \geq |q_1 - q_2|$, then min $\to p = \frac{1}{2} \Rightarrow (1+q_1 q_2)(1-3q_1 q_2) \geq 0$

$$q_1 q_2 \leq \frac{1}{3}$$

If $2|q_1 q_2| < |q_1 - q_2|$, then min $\to p=0$ or $p=1 \Rightarrow (1-q_1^2)(1-q_2^2) \geq 0$

$\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA $\iff$ $q_1 q_2 \leq \frac{1}{3}$.

depolarizing

Unital channels for qubits

$$\mathcal{E} = \begin{pmatrix} 1 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \end{pmatrix}, \qquad \overline{\mathcal{E}} = \begin{pmatrix} 1 & & & \\ & -\lambda_1 & & \\ & & -\lambda_2 & \\ & & & -\lambda_3 \end{pmatrix}.$$

__Lemma.__ Two-qubit channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA iff $\overline{\mathcal{E}_1} \otimes \mathcal{E}_2$ and $\mathcal{E}_1 \otimes \overline{\mathcal{E}_2}$ are positive maps.

__Proposition 1.__ $\mathcal{E}_1$ and $\mathcal{E}_2$ are unital qubit channels such that $\mathcal{E}_1^2$ and $\mathcal{E}_2^2$ are EB, then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA.

__Proposition 2__ A unital qubit channel $\mathcal{E}$ is 2-LEA if and only if channel $\mathcal{E}^2$ is EB, i.e., $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leq 1$

Points inside intersection of spere and tetrahedron correspond to 2-LEA

Proposition 1 gives sufficient condition. Surprisingly, $\mathcal{E}_1 \otimes \mathcal{E}_2$ can still be EA as demonstrated by the following example:

__Example:__ $\mathcal{E}_1 = \mathrm{diag}\left(1, \frac{1}{20}, \frac{1}{20}, 1\right)$, $\mathcal{E}_2 = \mathrm{diag}\left(1, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$.

Sufficient condition for $\mathcal{E}_1 \otimes \mathcal{E}_2$ not to be EA is

$$\vec{\lambda}_1 \cdot \vec{\lambda}_2 > 1.$$

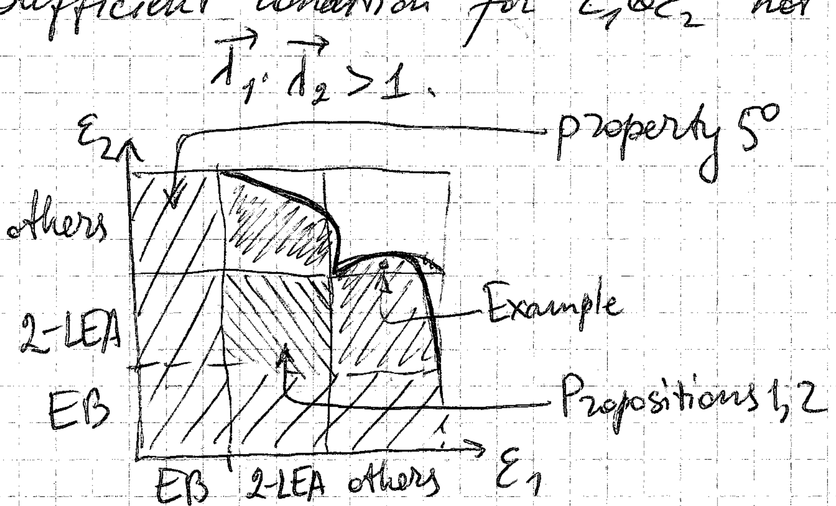


Figure: Entanglement annihilating two-qubit unital channels $\mathcal{E}_1 \otimes \mathcal{E}_2$.

BIBLIOGRAPHY: arXiv: 1110.3757 [quant-ph]