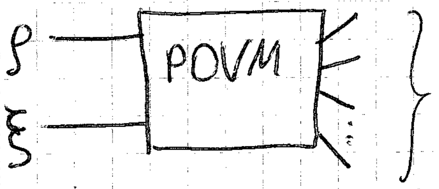
Episode 3 — Comparison of quantum states based on probabilities of outcomes



$\Big\}$ N outcomes

$$p_i = \mathrm{tr}\big[\underset{\uparrow}{E_i}(\rho \otimes \xi)\big]$$

POVM effect $\# i$

<u>Unambiguous comparison:</u>

possible answers:
- "same" ← only for locally-info-complete measurements with infinite precision
- "different"
- "I don't know"

Exercise:
POVM $= \{\Pi_+, \Pi_-\}$
$\mathrm{ran}(p_-^{same}) = ?$
$\mathrm{ran}(p_-^{diff}) = ?$

<u>Answer:</u> $p_-^{same} \in [0; \frac{1}{4}]$
$p_-^{diff} \in (0; \frac{1}{2}]$

<u>2-outcome POVM:</u>
(almost universal comparison)

$$E_1 = \begin{pmatrix} 1/2 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & 1/2 \end{pmatrix}, \quad E_2 = I - E_1.$$

All details and quantitative description see in

BIBLIOGRAPHY: arXiv: 1202.1015 [quant-ph]

Filippov, Ziman. Phys. Rev. A, <u>85</u>, 062301 (2012)

Episode 4 — <u>Quantum channels</u>

<u>Def.</u> Mapping $\mathcal{E}$ on $\mathcal{T}(\mathcal{H})$ is a <u>channel</u> if it is
- linear
- trace preserving
- <u>completely positive</u>

<u>Def.</u> A linear mapping $\mathcal{E}: \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ is completely positive if the mapping $\mathcal{E} \otimes \mathcal{I}_B$ is positive for <u>all</u> finite dimensional extensions $\mathcal{H}_B$.

<u>Matrix representation:</u> $\{\sigma_0, \dots, \sigma_{d^2-1}\}$ is a basis of orthogonal operators.

Any operator $A = \sum_j a_j \sigma_j$, $\quad a_j = \dfrac{\mathrm{tr}[\sigma_j^* A]}{\mathrm{tr}[\sigma_j^+ \sigma_j]}$

$$\mathcal{E}(A) = \sum_{jk} \underbrace{\frac{\mathrm{tr}[\sigma_j^* \mathcal{E}(\sigma_k)]}{\mathrm{tr}[\sigma_j^* \sigma_j]}}_{\mathcal{E}_{jk}} \frac{\mathrm{tr}[\sigma_k^* A]}{\mathrm{tr}[\sigma_k^* \sigma_k]} \sigma_j = \sum_{jk} \mathcal{E}_{jk} t_k \sigma_j$$

$$\vec{t} \longrightarrow \underset{\text{matrix}}{\|\mathcal{E}\|}\, \vec{t}$$

Matrix representation for qubits ($d=2$).

$\sigma_0 = I, \ \sigma_1 = \sigma_x, \ \sigma_2 = \sigma_y, \ \sigma_3 = \sigma_z$

$$\mathcal{E}_{jk} = \frac{1}{2} tr\left[\mathcal{E}(\sigma_j)\sigma_k\right]$$

$$\underset{4\times4}{\left(\mathcal{E}_{jk}\right)} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline t_1 & & & \\ t_2 & & \underset{3\times3}{T} & \\ t_3 & & & \end{array}\right)$$

$$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})$$

$$\mathcal{E}(\rho) = \frac{1}{2}(I + \vec{r}\,'\cdot\vec{\sigma})$$

$$\vec{r}\,' = T\cdot\vec{r} + \vec{t}$$

Def. Channel $\mathcal{E}$ is called *unital* if $\mathcal{E}(I) = I$

For qubits it means $\vec{t} = 0$.

Choi-Jamiołkowski representation

Informally: $\mathcal{E}: \mathcal{T}(\mathcal{H}_d) \to \mathcal{T}(\mathcal{H}_d)$ is represented by $d^2 \times d^2$ matrix that is associated with a state $\mathcal{S}(\mathcal{H}_{d^2})$

Choi theorem: Let $\mathcal{E}: \mathcal{M}_d \to \mathcal{M}_{d'}$ be a positive linear mapping. Then the following statements are equivalent:

(1) $\mathcal{E}$ is completely positive

(2) $\mathcal{E}$ is d-positive, i.e. $\mathcal{E}\otimes\mathcal{I}_d$ is a positive map

(3) the matrix

$$\Phi_{\mathcal{E}} = \begin{pmatrix} \mathcal{E}(|\varphi_1\rangle\langle\varphi_1|) & \cdots & \mathcal{E}(|\varphi_1\rangle\langle\varphi_d|) \\ \vdots & \ddots & \vdots \\ \mathcal{E}(|\varphi_d\rangle\langle\varphi_1|) & \cdots & \mathcal{E}(|\varphi_d\rangle\langle\varphi_d|) \end{pmatrix}$$

is positive, where $\{|\varphi_j\rangle\}$ is any orthonormal basis of d-dimensional Hilbert space and $\mathcal{E}(|\varphi_j\rangle\langle\varphi_k|)$ is an element of $\mathcal{M}_{d'}$.

Proof. (1) $\to$ (2) by definition of completely positive map.

(2) $\to$ (3) $\quad M = \sum_{jk} |\varphi_j\otimes\varphi_j\rangle\langle\varphi_k\otimes\varphi_k| \geq 0$

By positivity of $\mathcal{E}\otimes\mathcal{I}_d$ the matrix $M' = (\mathcal{E}\otimes\mathcal{I}_d)(M)$ is positive. Since $M' = \mathcal{E}(|\varphi_j\rangle\langle\varphi_k|)\otimes|\varphi_j\rangle\langle\varphi_k| = \Phi_{\mathcal{E}}$, the positivity of $\Phi_{\mathcal{E}}$ follows.

(3) → (1) $\psi_\ell \in \mathbb{C}^{d'} \otimes \mathbb{C}^{d}$ are eigenvectors (subnormalized) of the positive operator $\Phi_{\mathcal{E}}$, $\ell = 1, \dots, n \leq dd'$

$$\mathbb{C}^{d'} \otimes \mathbb{C}^{d} = \underbrace{\mathbb{C}^{d'} \oplus \dots \oplus \mathbb{C}^{d'}}_{d \text{ times}}$$

Let $P_j : \mathbb{C}^{d'} \otimes \mathbb{C}^{d} \to \mathbb{C}^{d'}_j$ be a projection onto the jth 'copy' of $\mathbb{C}^{d'}$. Then $\mathcal{E}[|\varphi_j\rangle\langle\varphi_k|] = P_j \Phi_{\mathcal{E}} P_k = \sum_\ell P_j |\psi_\ell\rangle\langle\psi_\ell| P_k =$

$$= \sum_\ell |P_j \psi_\ell\rangle\langle P_k \psi_\ell| = \sum_\ell |V_\ell \varphi_j\rangle\langle V_\ell \varphi_k| = \sum_\ell V_\ell |\varphi_j\rangle\langle\varphi_k| V_\ell^*$$

define $n \leq dd'$ operators $V_\ell : \mathbb{C}^d \to \mathbb{C}^{d'}$, $V_\ell \varphi_j = P_j \psi_\ell$

$$\mathcal{E}(X) = \sum_\ell V_\ell X V_\ell^*$$

completely positive map.

Q.E.D.

Choi-Jamiołkowski isomorphism:

$$\mathcal{J}: \quad \mathcal{E} \to \Omega_{\mathcal{E}} = (\mathcal{E} \otimes \mathcal{I})[P_+], \quad P_+ = \frac{1}{d} \sum_{j,k=1}^{d} |\varphi_j \otimes \varphi_j\rangle\langle\varphi_k \otimes \varphi_k|$$

maximally entangled state.

Informally: Channels $\mathcal{E}(\mathcal{H}_d)$ $\Longleftrightarrow$ States $\mathcal{S}(\mathcal{H}_{d^2})$

Exercise: Find the matrix representation and Choi matrix for the following qubit map: $\mathcal{E}[X] = q_0 \cdot X + q_1 \sigma_x X \sigma_x + q_2 \sigma_y X \sigma_y + q_3 \cdot \sigma_z \cdot X \cdot \sigma_z$.

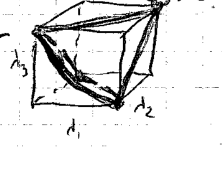
Find necessary and sufficient conditions for map $\mathcal{E}$ to be completely positive ~~and~~ trace preserving.

Answers: Matrix representation $(\mathcal{E}_{jk}) = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{array}\right)$, where

$\lambda_1 = q_0 + q_1 - q_2 - q_3$
$\lambda_2 = q_0 - q_1 + q_2 - q_3$, $\quad 1 = q_0 + q_1 + q_2 + q_3$ is trace preservation condition.
$\lambda_3 = q_0 - q_1 - q_2 + q_3$

Choi matrix
$$\Phi = \frac{1}{2}\begin{pmatrix} 1+\lambda_3 & 0 & 0 & \lambda_1+\lambda_2 \\ 0 & 1-\lambda_3 & \lambda_1-\lambda_2 & 0 \\ 0 & \lambda_1-\lambda_2 & 1-\lambda_3 & 0 \\ \lambda_1+\lambda_2 & 0 & 0 & 1+\lambda_3 \end{pmatrix} \geq 0 \Longleftrightarrow \begin{cases} |1+\lambda_3| \geq |\lambda_1+\lambda_2| \\ |1-\lambda_3| \geq |\lambda_1-\lambda_2| \end{cases} \Longleftrightarrow \begin{array}{l} q_0 \geq 0, \\ q_1 \geq 0, \\ q_2 \geq 0, \\ q_3 \geq 0 \end{array}$$

$\mathcal{E}$ is a valid channel if point $(\lambda_1, \lambda_2, \lambda_3)$ is inside ~~te~~tetrahedron



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